Exercise set 1 (Languages and problems)

1. Let α and β be strings over an alphabet Σ. Show that:
(a) $(\alpha\beta)^R = \beta^R\alpha^R$.
(b) $(\alpha^R)^R = \alpha$.
(c) $(\alpha^n)^R = (\alpha^R)^n$ for every $n \in \mathbb{Z}_+$.

2. Let L be a language over an alphabet Σ. Prove or disprove: $L^+ = L^* \setminus \{\epsilon\}$.

3. **(a)** Prove or disprove: $baaa \in a^*b^*c^*a^*$, $baca \in a^*b^*c^*a^*$ and $abcd \in (a(cd)^*b)^*$.
(b) What are the elements of the language $(a^*b^*) \cap (b^*c^*)$?

4. **(a)** Show that the regular expression $(a^* \cup b^*)^*$ represents the language $\{a,b\}^*$ and so do $(a \cup b)^*$, $a^*(ba^*)^*$ and $b^*(ab^*)^*$.

(b) Show that each of the following regular expressions represents the language $\{a,b\}^+$ (i.e., the language of all non-empty strings over $\{a,b\}$): $(a \cup b)(a \cup b)^*$, $(a(a \cup b)^*) \cup (b(a \cup b)^*)$, $(aa^*(ba^*)^*) \cup (bb^*(ab^*)^*)$ and $(a^*b(a \cup b)^*) \cup (b^*a(a \cup b)^*)$.

5. Let α and β be regular expressions over an alphabet Σ. Prove or disprove the following statements:
(a) $\mathcal{L}((\alpha \cup \beta)^*) = \mathcal{L}(\alpha^* \cup \beta^*)$.
(b) $\mathcal{L}((\alpha \cup \beta)^*) = \mathcal{L}((\alpha^* \cup \beta^*)^*)$.
(c) $\mathcal{L}((\alpha\beta)^*) = \mathcal{L}(\alpha^*\beta^*)$.
(d) $\mathcal{L}((\alpha\beta)^*) = \mathcal{L}((\alpha^*\beta^*)^*)$.
(e) $\mathcal{L}(\alpha^*) = \mathcal{L}((\alpha^*)^*)$.
(f) $\mathcal{L}(\alpha) = \mathcal{L}(\epsilon \cup \alpha)$.

6. Write regular expressions for the following languages:
(a) The set of binary expressions of all odd positive integers.
(Assume that in a binary expression of a positive integer no leading 0's are allowed.)
(b) The set of all strings over $\{a,b,c\}$ containing at least one a and at least one b.
(c) The set of strings over $\{a,b\}$ whose fifth symbol from the right end is a.
(d) The set of strings over $\{a,b\}$ in which the number of occurrences of a is a multiple of 5.
(e) The set of strings over $\{a,b\}$ in which the number of occurrences of a is $\leqslant 5$.

7. Give English descriptions of the languages represented by the regular expressions:
(a) $(b \cup \epsilon)(aa^*b)^*a^*$
(b) $(a \cup ba)^*(b \cup \epsilon)$
(c) $(a \cup ba)^*b^*$

*** 8.** **(a)** Prove that if L is a regular language, then so is the language L' consisting of all suffixes of all strings of L. Let us call L' the suffix language of L.

(b) Find a regular expression representing the suffix language of the language defined by the regular expression $a(a \cup b)^*$ over the alphabet $\{a,b\}$.

*** 9.** **(a)** A regular expression is said to be in *disjunctive normal form (DNF)*, if it is of the form $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ for some $n \in \mathbb{N}$, where none of the α_i's contains an occurrence of $\cup$. Show that every regular language can be represented by a regular expression in DNF. (**Hint:** Exercise 4(a).)

(b) We say that a regular expression β is a disjunctive normal form (DNF) of a regular expression α, if β is in DNF and $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$. Find a DNF of each of the regular expressions (over the alphabet $\{a,b,c,d\}$): $(a \cup b \cup c)^*$, $(a \cup b)^*(c \cup d)$, $(a \cup b)^*(c \cup d)^*$ and $(a \cup b)^* \cup (c \cup d)^*$.