Exercise set 0 (Preliminaries)

1. Which of the relations $=, <, \leqslant, >, \geqslant, \neq$ are equivalence relations on $\mathbb{N}$ ($\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{R}$)? Partial orders? Total orders? Well-orders?

2. Find the flaw in the following proof.

Theorem All horses are of the same color.

Proof Let there be n horses. We proceed by induction on n. If $n = 1$, there is nothing to prove. So assume that $n > 1$ and that the theorem holds for any group of $n - 1$ horses. From the given n horses discard one, say the first one. Then all the remaining $n - 1$ horses are of the same color by the induction hypothesis. Now put the first horse back and discard another, say the last one. Then the first $n - 1$ horses have the same color again by the induction hypothesis. So all the n horses must have the same color as the ones that were not discarded either time. •

3. **(a)** Let $f : A \to B$ and $g : B \to A$ be two functions. If $g \circ f = \mathrm{id}_A$, show that f is injective and g is surjective.

(b) Show that a function $f : A \to B$ is a bijection, if and only if there exists a unique $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case we write $g = f^{-1}$ and $f = g^{-1}$.

4. **(a)** Show that each of the following sets is equinumerous with $\mathbb{N}$ (and is hence countably infinite): $\mathbb{N}_{\text{odd}}, \mathbb{N}_{\text{even}}, \mathbb{Z}, \mathbb{Z}_{\text{odd}}, \mathbb{Z}_{\text{even}}$.

* **(b)** Show that $\mathbb{R}$ is equinumerous with $\mathbb{C}$.

5. Show that the function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $(i, j) \mapsto 1 + \frac{1}{2}\left[(i + j)^2 - i - 3j\right]$ is a bijection.

6. Prove that the sets $\mathbb{Z}[X], \mathbb{Z}[X_1, \ldots, X_n]$ are countable.

7. **(a)** Let A and B be two sets of finite cardinalities m and n respectively. Show that the number of functions $A \to B$ is n^m.

(b) Show that for any set A there is a bijection between the power set A and the set of all functions $A \to \{0, 1\}$.

* **8.** Show that no set A is equinumerous with its power set 2^A.

9. Give an example of a non-polynomial function $f(n)$ that has the same rate of growth as a polynomial function, say n^d.

10. Show that the logarithm function $\log n$ is in $\mathrm{O}(n^\epsilon)$ for every $\epsilon > 0$. Deduce that the log function has strictly smaller rate of growth than n^ϵ for any $\epsilon > 0$.

11. **(a)** Let $f_1(n) \asymp g_1(n)$ and $f_2(n) \asymp g_2(n)$. Show that $f_1(n) + f_2(n) \asymp g_1(n) + g_2(n)$ and $f_1(n)f_2(n) \asymp g_1(n)g_2(n)$.

(b) Prove that the converse of Part (a) is not true in general, that is, $f_1(n) + f_2(n) \asymp g_1(n) + g_2(n)$ (or $f_1(n)f_2(n) \asymp g_1(n)g_2(n)$) does not necessarily imply that $f_1(n) \asymp g_1(n)$ and $f_2(n) \asymp g_2(n)$ (or $f_1(n) \asymp g_2(n)$ and $f_2(n) \asymp g_1(n)$).

12. Let $f, g : \mathbb{N} \to \mathbb{R}_+$ be monotonically increasing functions that tend to ∞ as $n \to \infty$. Show that:

(a) $f(n) \in \mathrm{O}(g(n))$, if and only if there exist real constants $c > 0$ and $d \geqslant 0$ with $f(n) \leqslant cg(n) + d$ for all $n \in \mathbb{N}$.

(b) $f(n) \asymp g(n)$, if and only if there exist real constants $c_1, c_2 > 0$ and $d_1, d_2 \geqslant 0$ with $f(n) \leqslant c_1 g(n) + d_1$ and $g(n) \leqslant c_2 f(n) + d_2$ for all $n \in \mathbb{N}$.

13. Let $g : \mathbb{N} \to \mathbb{R}_+$ be a nowhere-zero function (i.e., $g(n) \neq 0$ for all $n \in \mathbb{N}$). We define $\mathrm{o}(g(n))$ to be the set of all functions $f : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{n\to\infty} f(n)/g(n) = 0$. Prove that $f(n) \in \mathrm{o}(g(n))$, if and only if $f(n) \in \mathrm{O}(g(n))$ and $g(n) \notin \mathrm{O}(f(n))$.