MTH 215 Number Theory : Second Mid Semester Examination (Exercise Set A)

Total marks: 25 | March 17, 2002 | Time: $1+\epsilon$ hours

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1. Find all the solutions of the congruence: $x^2 + x - 1 \equiv 0 \pmod{437}$. (**Remark:** $437 = 19 \times 23$). **(5)**

2. Let p be a prime of the form $4k+1$. Find the sum of the quadratic residues (modulo p) in the set $\{1, \dots, p-1\}$. (**Hint:** Look at $\left(\frac{-1}{p}\right)$.) **(3)**

3. Let p be a prime of the form $4k+1$ and let a and b be positive integers with a odd and $a^2 + b^2 = p$. Show that $\left(\frac{a}{p}\right) = 1$. (**Hint:** Call Gauss.) **(3)**

4. Let $n \in \mathbb{N}$. Show that:

(a) The equation $\sigma(x) = n$ may not have a solution (where x is a positive integer). **(2)**

(b) If $n \geqslant 2$, there exists an infinite number of positive integers x satisfying $\delta(x) = n$. **(2)**

5. Let f be a complex-valued multiplicative arithmetical function and let k be a non-negative integer.

(a) For which values of k is the function $F(n) := \sum_{d|n} f(d)^k$ necessarily multiplicative? **(3)**

(b) Is the function $G(n) := \prod_{d|n} f(d)^2$ necessarily multiplicative? **(3)**

(**Remark:** Here "necessarily" means "irrespective of f".)

6. Find the explicit formula for the sequence u_n, $n \in \mathbb{N}$, which satisfies: $u_1 = 1$, $u_2 = 1$ and $u_n = 6(u_{n-2} + \cdots + u_1) + 1$ for $n \geqslant 3$. (**Hint:** You may reduce the problem to the case of a (homogeneous) linear recurrence of order two.) **(4)**

MTH 215 Number Theory : Second Mid Semester Examination (Exercise Set B)

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1. Find all the solutions of the congruence: $x^2 + x + 1 \equiv 0 \pmod{437}$. (**Remark:** $437 = 19 \times 23$). **(5)**

2. Let p be a prime of the form $4k+1$. Find the sum of the quadratic non-residues (modulo p) in the set $\{1, \dots, p-1\}$. (**Hint:** Look at $\left(\frac{-1}{p}\right)$.) **(3)**

3. Let p be a prime of the form $4k+1$ and let a and b be positive integers with a odd and $a^2 + b^2 = p$. Show that $\left(\frac{a}{p}\right) = 1$. (**Hint:** Call Gauss.) **(3)**

4. Let $n \in \mathbb{N}$. Show that:

(a) $\delta(x) = n$ for some positive integer x. **(2)**

(b) There exists only a finite number of positive integers x satisfying $\sigma(x) = n$. **(2)**

5. Let f be a complex-valued multiplicative arithmetical function and let k be a non-negative integer.

(a) For which values of k is the function $F(n) := \prod_{d|n} f(d)^k$ necessarily multiplicative? **(3)**

(b) Is the function $G(n) := \sum_{d|n} f(d)^2$ necessarily multiplicative? **(3)**

(**Remark:** Here "necessarily" means "irrespective of f".)

6. Find the explicit formula for the sequence u_n, $n \in \mathbb{N}$, which satisfies: $u_1 = 1$, $u_2 = 1$ and $u_n = 12(u_{n-2} + \cdots + u_1) + 1$ for $n \geqslant 3$. (**Hint:** You may reduce the problem to the case of a (homogeneous) linear recurrence of order two.) **(4)**