MTH 215 Number Theory

First Mid Semester Examination (Exercise set A)

Total marks: 25 | February 6, 2002 | Time: $1 + \epsilon$ hours

---

Name: **Roll Number:**

---

1. Let $a, b \in \mathbb{N}$, $b < a$ and $\gcd(a, b) = 1$. Show that $\binom{a}{b}$ is divisible by a and that $\binom{a-1}{b-1}$ is divisible by b. **(2)**

2. Let $a, b, c, n \in \mathbb{N}$, $\gcd(a, b) = 1$ and $ab = c^n$. Prove that $a = x^n$ and $b = y^n$ for some $x, y \in \mathbb{N}$. **(3)**

3. Let p be an odd prime number and $M_p = 2^p - 1$ the Mersenne number corresponding to p. If q is a prime divisor of M_p, show that $q \equiv 1 \pmod{2p}$. **(3)**

4. Compute all the solutions of the congruence $x^2 + x + 1 \equiv 0 \pmod{91}$. **(4)**

5. How many solutions does the congruence $x^{64} + x^{32} + 1 \equiv 0 \pmod{97}$ have? **(3)**

6. Let $m, n \in \mathbb{N}$ with m odd. Show that $\gcd(2^m - 1, 2^n + 1) = 1$. **(4)**

7. The sequence of Fibonacci numbers is defined as $F_0 := 0$, $F_1 := 1$ and $F_m := F_{m-1} + F_{m-2}$ for $m \geqslant 2$. The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

(a) Show that $F_m = \frac{1}{\sqrt{5}}(\rho^m - \bar{\rho}^m)$ for all $m \in \mathbb{Z}_+$, where $\rho = \frac{1+\sqrt{5}}{2}$ and $\bar{\rho} = \frac{1-\sqrt{5}}{2}$. **(3)**

(b) Let $n \in \mathbb{N}$. Show that $n | F_m$ for some $m \in \mathbb{N}$. **(3)**

MTH 215 Number Theory

First Mid Semester Examination (Exercise set B)

Total marks: 25 | February 6, 2002 | Time: $1+\epsilon$ hours

---

Name: **Roll Number:**

---

1. Let $r \in \mathbb{N}$, $m_1, \ldots, m_r \in \mathbb{Z}_+$, $m = \sum_{i=1}^r m_i$ and q a prime number satisfying $\max(m_1, \ldots, m_r) < q \leqslant m$. Show that q divides the multinomial coefficient $\binom{m}{m_1, \ldots, m_r} = \frac{m!}{m_1! \cdots m_r!}$. **(2)**

2. Let $a, b, m, n \in \mathbb{N}$, $\gcd(m,n) = 1$ and $a^m = b^n$. Show that $a = x^n$ and $b = x^m$ for some $x \in \mathbb{N}$. **(3)**

3. Let $m \in \mathbb{N}$ and $p \in \mathbb{P}$. Show that if $p|\phi(m)$ and $p \nmid m$, then $q \equiv 1 \pmod{p}$ for some prime factor q of m. **(3)**

4. Compute all the solutions of the congruence $x^2 + x - 1 \equiv 0 \pmod{121}$. **(4)**

5. How many solutions does the congruence $x^{50} \equiv 10 \pmod{101}$ have? **(3)**

6. Let $p \in \mathbb{P}$ and $k \in \mathbb{Z}$. Show that $1^k + 2^k + \cdots + (p-1)^k \equiv \begin{cases} 0 \pmod{p} & \text{if } (p-1) \nmid k, \\ -1 \pmod{p} & \text{if } (p-1)|k. \end{cases}$ **(4)**

7. The sequence of Fibonacci numbers is defined as $F_0 := 0$, $F_1 := 1$ and $F_m := F_{m-1} + F_{m-2}$ for $m \geqslant 2$. The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Show that:

(a) $F_{m+n} = F_m F_{n-1} + F_{m+1} F_n$ for all $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. **(2)**

(b) $\gcd(F_m, F_{m+1}) = 1$ for all $m \in \mathbb{Z}_+$. **(2)**

(c) $\gcd(F_m, F_n) = F_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$. **(2)**
