RECURSION

CS10003: PROGRAMMING AND DATA STRUCTURES

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Recursion

A process by which a function calls itself repeatedly.

- Either directly.
    - F calls F.
- Or cyclically in a chain.
    - F calls G, G calls H, and H calls F.

Used for repetitive computations in which each action is stated in terms of a previous result.

fact(n) = n * fact (n-1)

Basis and Recursion

For a problem to be written in recursive form, two conditions are to be satisfied:

- It should be possible to express the problem in recursive form.
- The problem statement must include a stopping condition

```
fact(n)  =  1,                  if  n = 0        /* Stopping criteria */
         =  n * fact(n – 1),    if  n > 0        /* Recursive form */
```

Examples:

- **Factorial:**

  fact(0) = 1

  fact(n) = n * fact(n − 1), if n > 0

- **GCD (assume that m and n are non-negative and m ≥ n):**

  gcd (m, 0) = m

  gcd (m, n) = gcd (n, m%n) , if n > 0

- **Fibonacci sequence (0,1,1,2,3,5,8,13,21,…)**

  fib (0) = 0

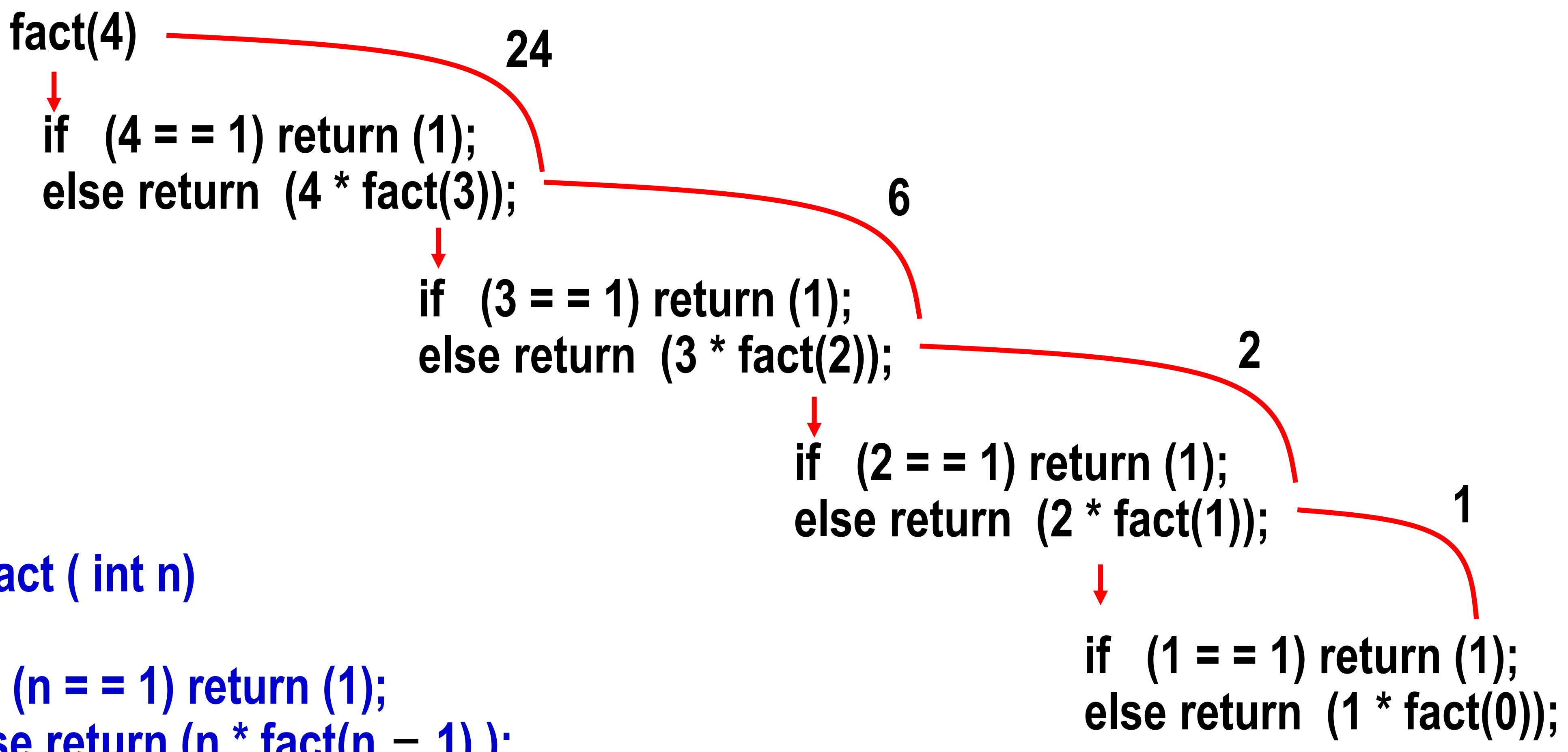
  fib (1) = 1

  fib (n) = fib (n − 1) + fib (n − 2), if n > 1

Example 1 :: Factorial

```
int  fact ( int n)
{
    if   (n = = 1)
        return (1);
    else
        return  (n * fact(n – 1));
}
```

Example 1 :: Factorial Execution



```
fact(4)
  if   (4 = = 1) return (1);
  else return  (4 * fact(3));
                    if   (3 = = 1) return (1);
                    else return  (3 * fact(2));
                                      if   (2 = = 1) return (1);
                                      else return  (2 * fact(1));
                                                     if   (1 = = 1) return (1);
                                                     else return  (1 * fact(0));
```

```
int  fact ( int n)
{
   if   (n = = 1) return (1);
   else return (n * fact(n – 1) );
}
```

Example 2 :: Fibonacci number

Fibonacci number f(n) can be defined as:

$$f(0) = 0$$

$$f(1) = 1$$

$$f(n) = f(n-1) + f(n-2), \quad \text{if } n > 1$$

- The successive Fibonacci numbers are:

  0, 1, 1, 2, 3, 5, 8, 13, 21, …..

```
int   f (int n)
{
     if  (n  < 2)   return (n);
     else  return ( f(n – 1) + f(n – 2) );
}
```

Tracing Execution

```
int   f (int n)
{
    if  (n  < 2)   return (n);
    else  return ( f(n – 1) + f(n – 2) );
}
```

$$\begin{tikzcd}
 & & & & f(4) \arrow[dll] \arrow[dr] & & \\
 & & f(3) \arrow[dl] \arrow[dr] & & & f(2) \arrow[dl] \arrow[dr] & \\
 & f(2) \arrow[dl] \arrow[dr] & & f(1) & f(1) & & f(0) \\
f(1) & & f(0) & & & &
\end{tikzcd}$$

How many times is the function called when evaluating f(4) ?

called 9 times

Inefficiency:

- Same thing is computed several times.

Some points to note

Every recursive program can also be written without recursion

- Tail Recursion: Last thing a recursive function does is making a single recursive call (of itself) at the end.
- Easy to replace tail recursion by a loop.
- In general, removal of recursion may be a very difficult task (even if you have your own recursion stack).

Recursion can be helpful in many situations

- Better readability
- Ease of programming
- Sometimes, recursion gives best-possible or best-known algorithms to solve problems

Recursion can also be a killer

- You solve the same subproblem multiple times (Example: Fibonacci numbers)
- Every recursive call incurs a (small) overhead

Use recursion with caution

Example of tail recursion

Not a tail recursion:

```
int sum1 ( int n )
{
        if (n == 0) return 0;
        return n + sum1(n–1);
}
```

Tail recursion:

```
int sum2 ( int n, int partialsum )
{
        if (n == 0) return partialsum;
        return sum2(n – 1, n + partialsum);
}
```

Call from main() as:

```
scanf("%d", &N);
s = sum2(N, 0);
```

Equivalent iterative function:

```
int sum3 ( int n )
{
        int partialsum = 0;
        while (n > 0) {
                partialsum = n + partialsum;
                n = n – 1;
        }
        return partialsum;
}
```

Important things to remember

- Think how the current problem can be solved if you can solve exactly the same problem on one or more smaller instance(s).
- Do NOT think how the problem will be solved on smaller instances, just call the function recursively and assume that the recursive calls do their jobs correctly.
- Do NOT forget to include the base cases to solve the problem on *smallest* instances.
- This is basically mathematical induction applied to programming.

- When you write a recursive function
    - First, write the terminating/base condition
    - Then, write the rest of the function
    - Always double-check that you have both

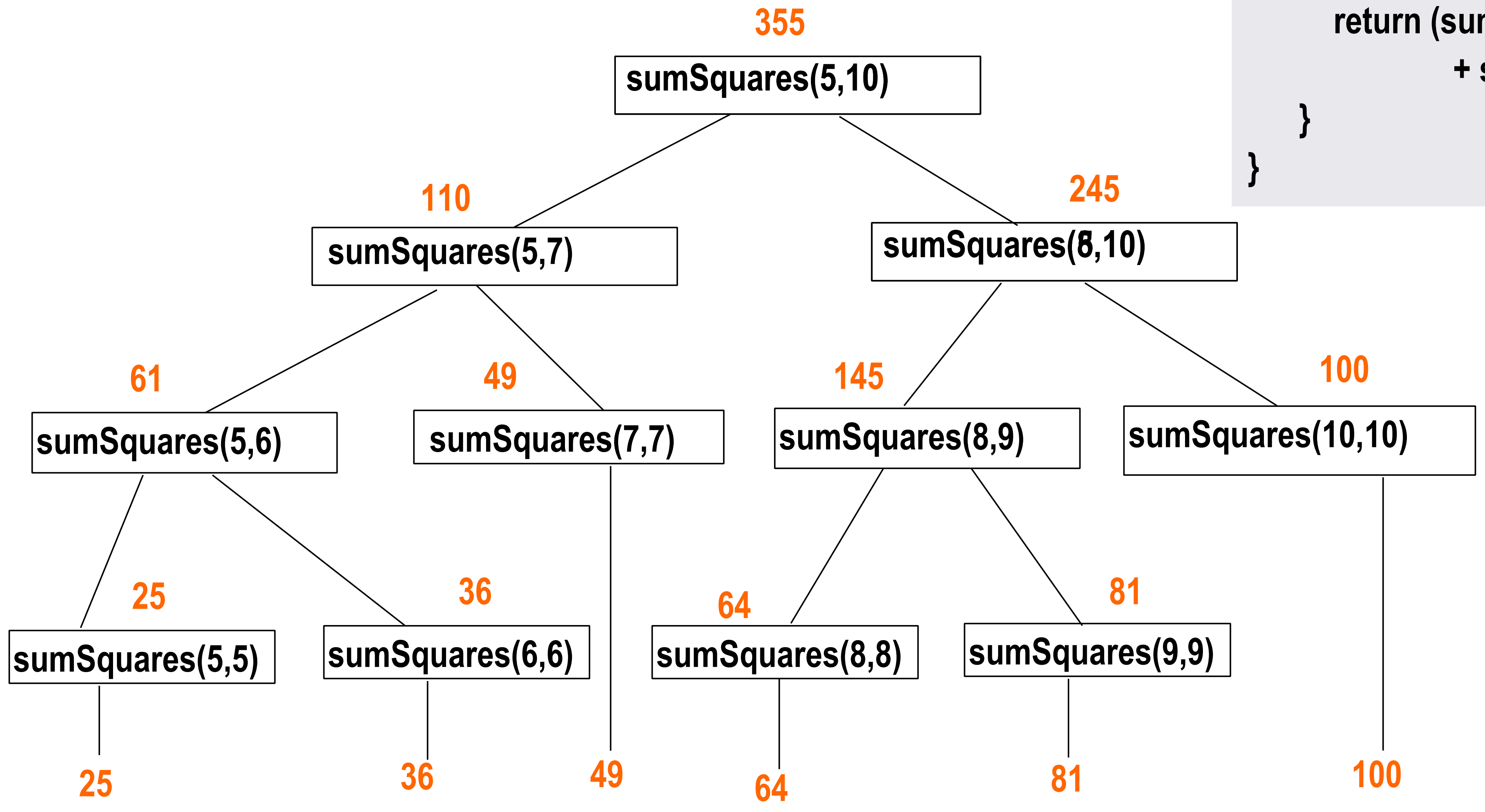
Example: Sum of Squares

Write a function that takes two integers m and n as arguments, and computes and returns the sum of squares of every integer in the range [m:n], both inclusive.

```
int sumSquares (int m, int n)
{
    int middle ;
    if (m == n) return(m*m);
    else
    {
        middle = (m+n)/2;
        return (sumSquares(m,middle)  + sumSquares(middle+1,n));
     }
}
```

Annotated Call Tree

```
int sumSquares (int m, int n)
{
    int middle ;
    if (m == n) return(m*m);
    else {
        middle = (m+n)/2;
        return (sumSquares(m,middle)
                + sumSquares(middle+1,n));
     }
}
```



Example: Printing the digits of an integer in reverse

Print the last digit, then print the remaining number in reverse

- Ex: If integer is 743, then reversed is print 3 first, then print the reverse of 74

```
void printReversed ( int i )
{
    if (i < 10)   {
      printf(“%d\n”, i); return;
    }
    else {
       printf(“%d”, i%10);
       printReversed(i/10);
    }
}
```

Example: Printing your name in reverse

```
#include <stdio.h>

void readandprint ()
{
        char c;

        scanf("%c", &c);
        if (c == '\n') return;
        readandprint();
        printf("%c", c);
}

int main ()
{
        printf("Enter your name and hit return: ");
        readandprint();
        printf("\n");
}
```

Output

```
Enter your name and hit return: Jane Doe
eoD enaJ
```

Exercise: Rewrite this code so that the output looks as follows:

```
Enter your name and hit return: Jane Doe
Your name in reverse: eoD enaJ
```

Counting Zeros in a Positive Integer

Check last digit from right

- If it is 0, number of zeros = 1 + number of zeroes in remaining part of the number
- If it is non-0, number of zeros = number of zeroes in remaining part of the number

```
int zeros (int number)
{
    if(number < 10) return 0;
    if (number % 10 == 0)
                return( 1 + zeros(number/10) );
    else
      return( zeros(number/10) );
}
```

Common Errors in Writing Recursive Functions

Non-terminating Recursive Function (Infinite recursion)

- No base case

```
int badFactorial(int x) {
  return x * badFactorial(x-1);
}
```

- The base case is never reached

```
int badSum2(int x)
{
  if(x==1) return 1;
  return(badSum2(x--));
}
```

```
int anotherBadFactorial(int x) {
  if(x == 0)
    return 1;
  else
    return x*(x-1)*anotherBadFactorial(x-2);
    // When x is odd, base case is never reached!!
}
```

Common Errors in Writing Recursive Functions

Mixing up loops and recursion

```
int anotherBadFactorial (int x) {
        int i, fact = 0;
        if (x == 0) return 1;
        else {
                for (i=x; i>0; i=i-1) {
                        fact = fact + x*anotherBadFactorial(x-1);
                }
                return fact;
        }
}
```

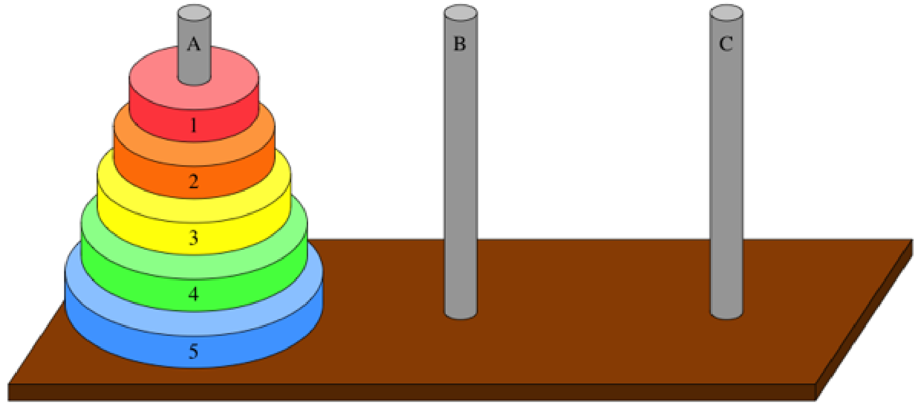
In general, if you have recursive function calls within a loop, think carefully if you need it.

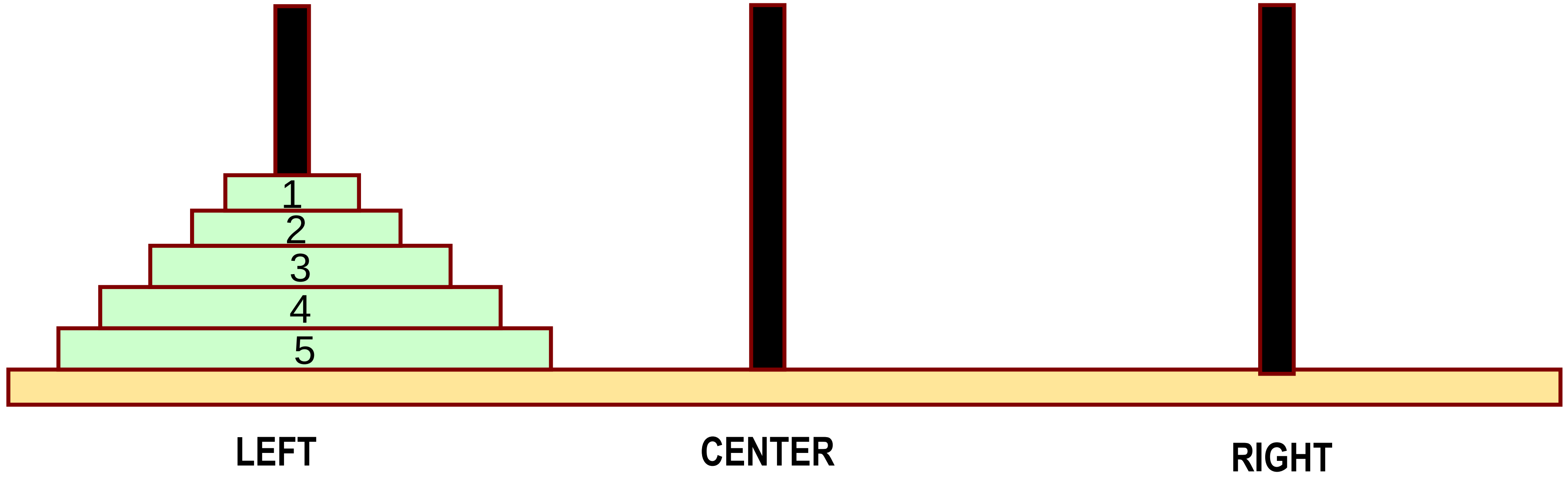
Most recursive functions you will see in this course will not need this

Example :: Towers of Hanoi Problem

The problem statement:

- Initially all the disks are stacked on the LEFT pole.
- Required to transfer all the disks to the RIGHT pole.
    - Only one disk on the top can be moved at a time.
    - A larger disk cannot be placed on a smaller disk.
- CENTER pole is used for temporary storage of disks.

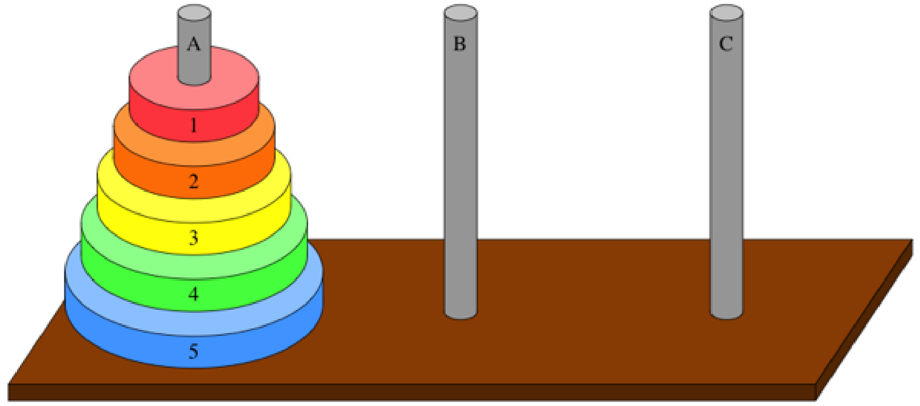




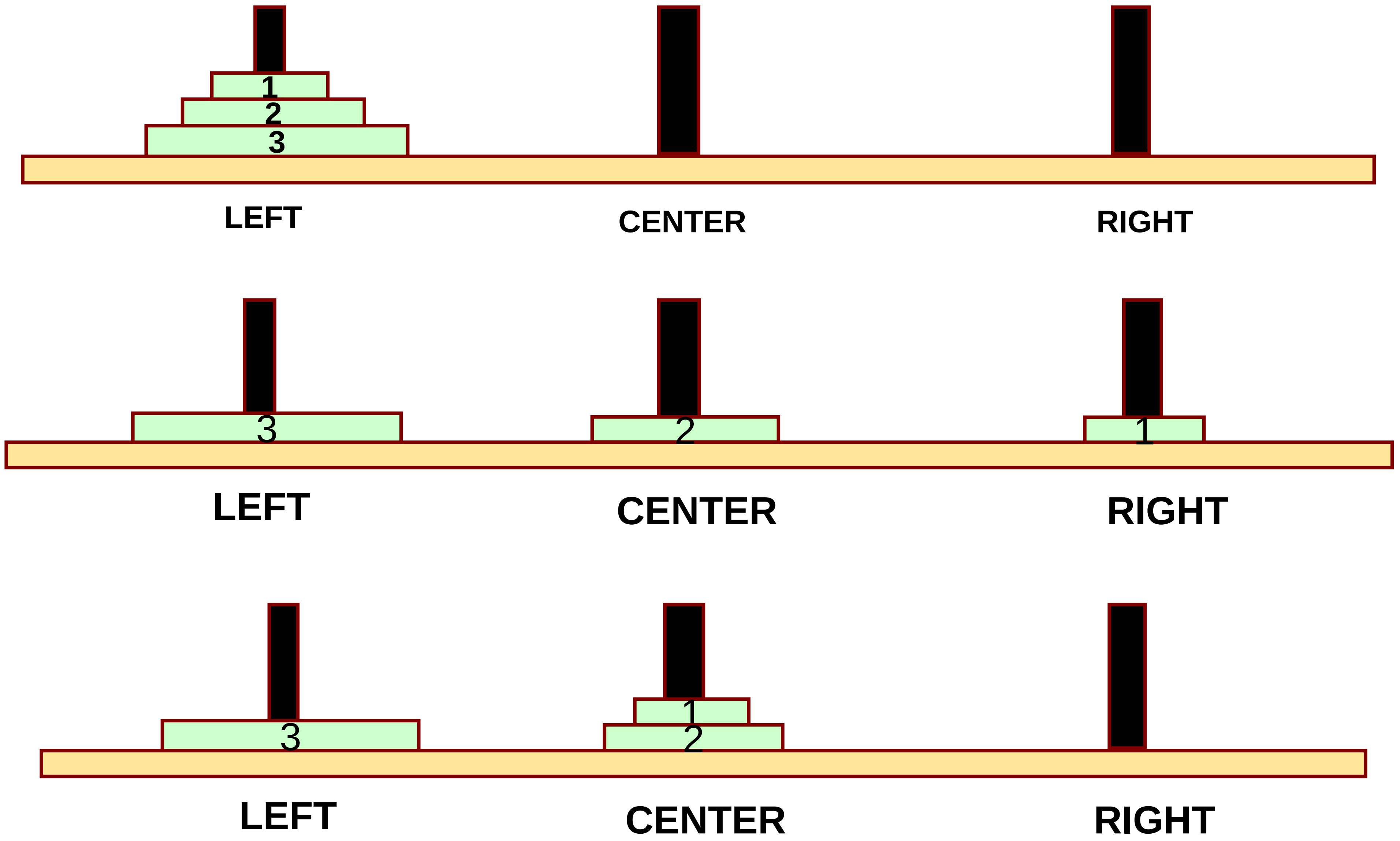
Recursive Formulation

Recursive statement of the general problem of n disks.

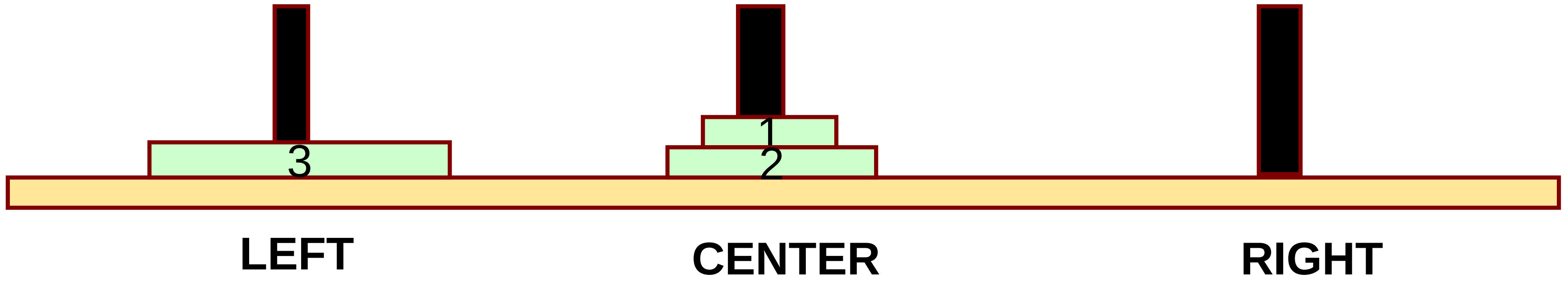
- **Step 1:**
  - **Move the top (n-1) disks from LEFT to CENTER.**
- **Step 2:**
  - **Move the largest disk from LEFT to RIGHT.**
- **Step 3:**
  - **Move the (n-1) disks from CENTER to RIGHT.**

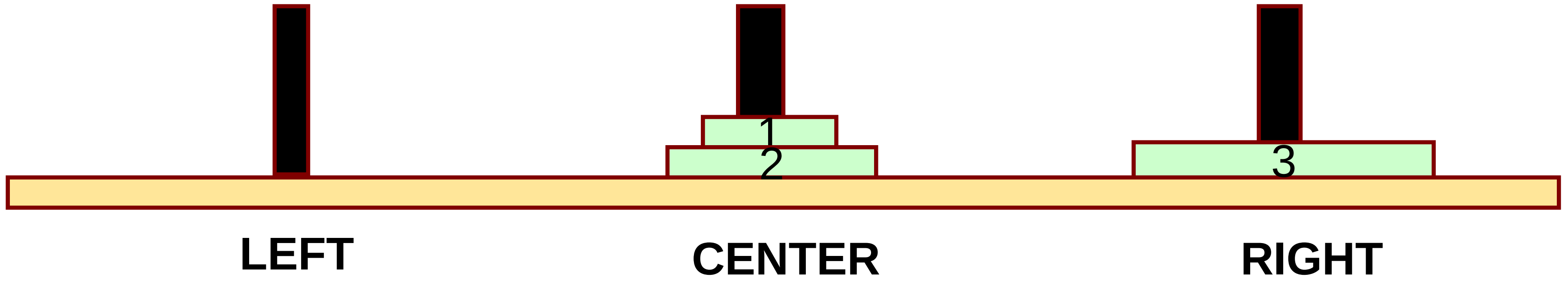


Phase-1: Move top n – 1 from LEFT to CENTER

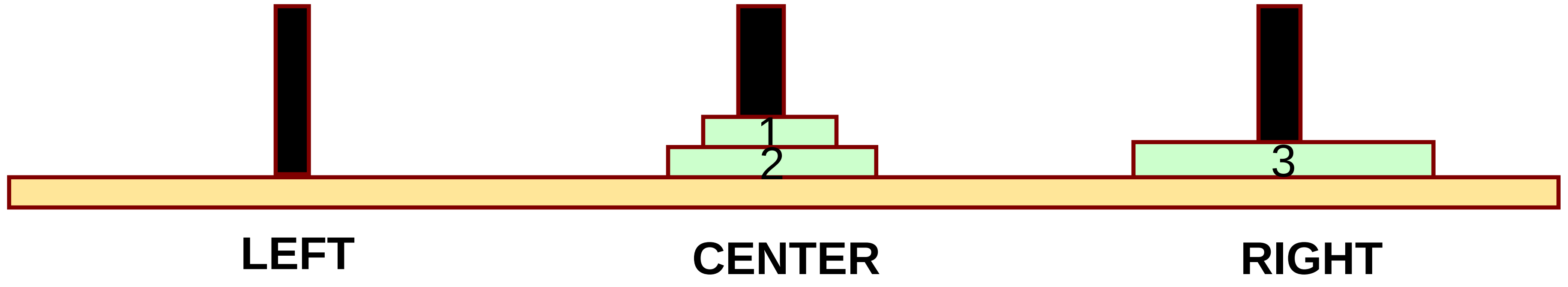


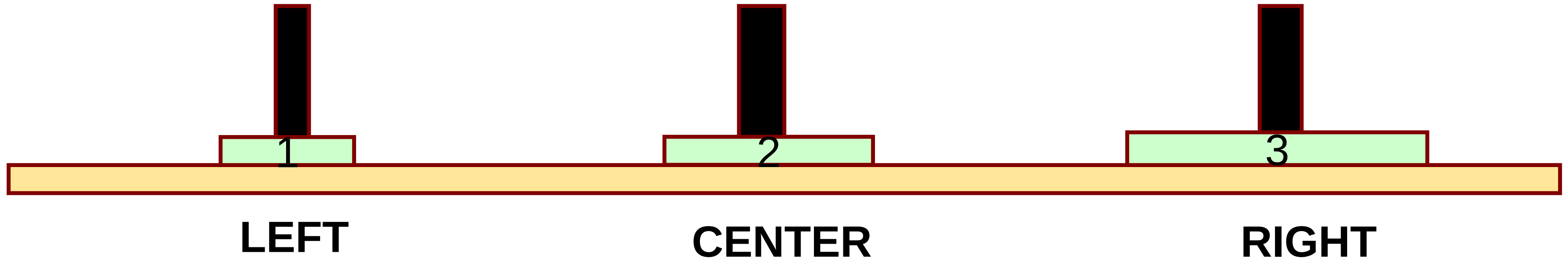
Phase-2: Move the nth disk from LEFT to RIGHT

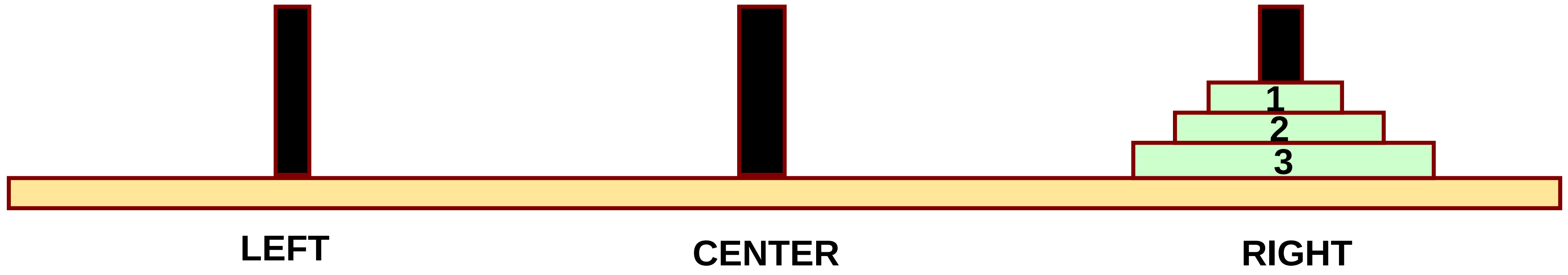




Phase-3: Move top n – 1 from CENTER to RIGHT



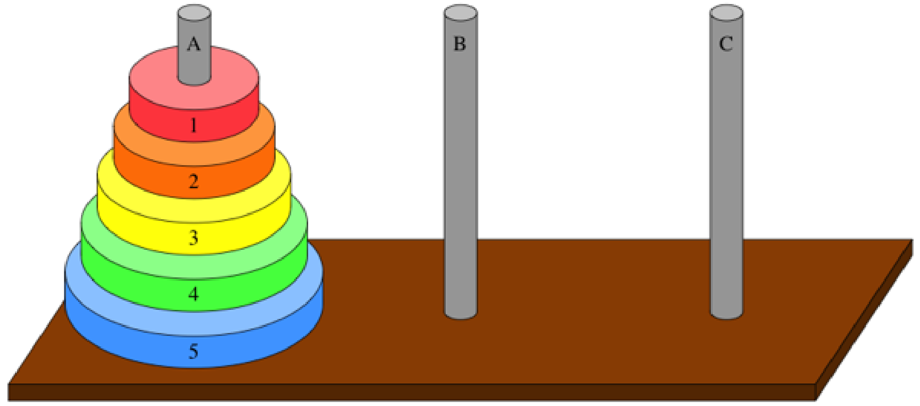


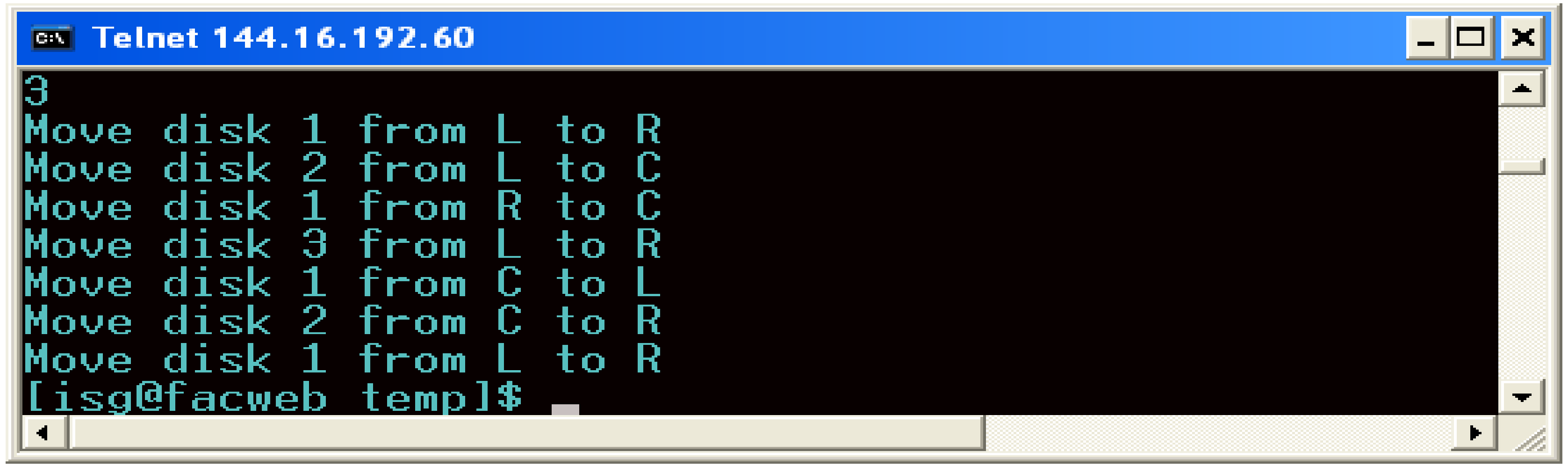


```
#include  <stdio.h>
void  transfer (int n, char from, char to, char temp);

int main( )
{          int  n;  /* Number of disks */
           scanf (“%d”, &n);
           transfer (n, ‘L’, ‘R’, ‘C’);
           return 0;
}

void  transfer (int n, char from, char to, char temp)
{
           if  (n > 0)  {
                     transfer  (n-1, from, temp, to);
                     printf (“Move disk %d from %c to %c \n”, n, from, to);
                     transfer (n-1, temp, to, from);
           }
           return;
}
```

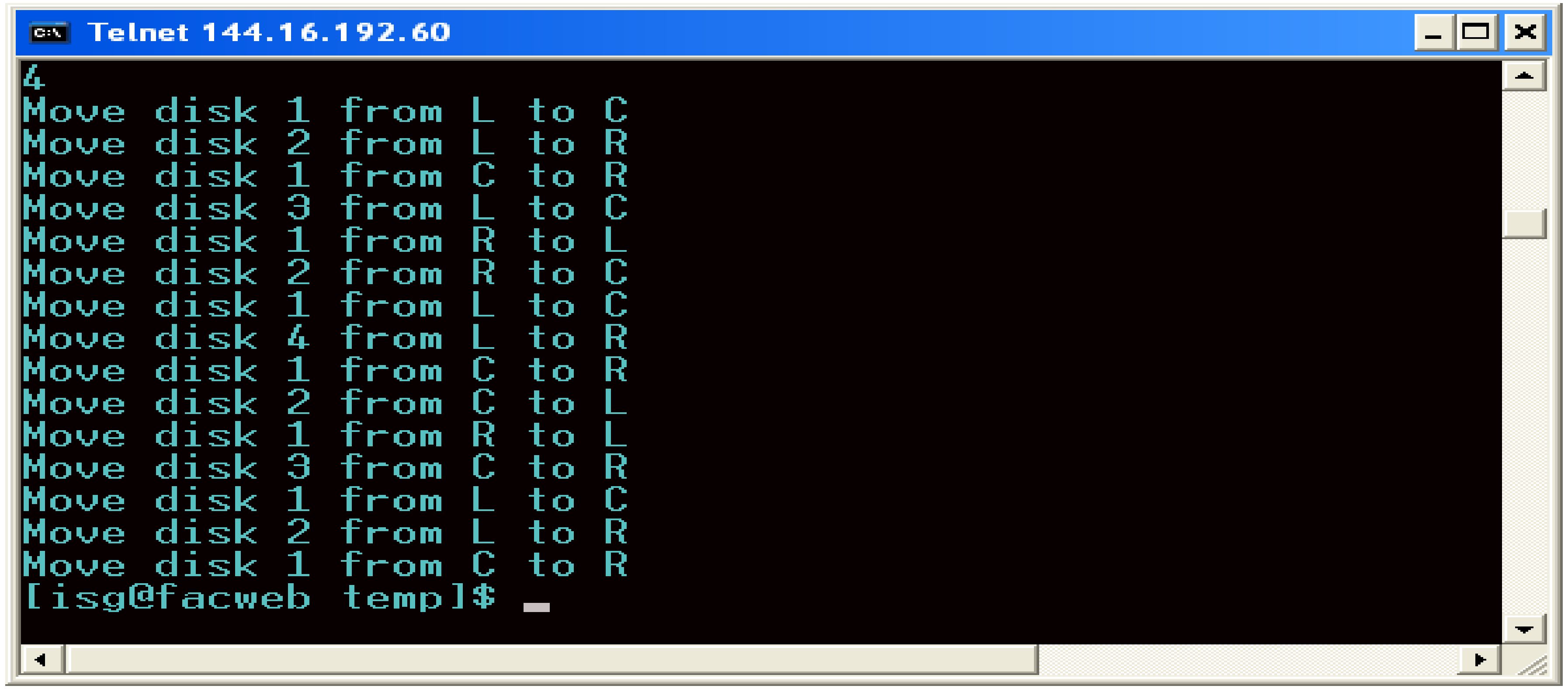




Telnet 144.16.192.60

```
3
Move disk 1 from L to R
Move disk 2 from L to C
Move disk 1 from R to C
Move disk 3 from L to R
Move disk 1 from C to L
Move disk 2 from C to R
Move disk 1 from L to R
[isg@facweb temp]$
```

With 3 discs



Telnet 144.16.192.60

```
4
Move disk 1 from L to C
Move disk 2 from L to R
Move disk 1 from C to R
Move disk 3 from L to C
Move disk 1 from R to L
Move disk 2 from R to C
Move disk 1 from L to C
Move disk 4 from L to R
Move disk 1 from C to R
Move disk 2 from C to L
Move disk 1 from R to L
Move disk 3 from C to R
Move disk 1 from L to C
Move disk 2 from L to R
Move disk 1 from C to R
[isg@facweb temp]$
```

With 4 discs

Recursion versus Iteration

Repetition

- Iteration: explicit loop
- Recursion: repeated nested function calls

Termination

- Iteration: loop condition fails
- Recursion: base case recognized

Both can have infinite loops

Balance

- Understand the benefits / penalties of recursion in terms of
    - Ease of implementation
    - Readability
    - Performance degradation / performance enhancement
- Take an educated decision

More Examples

What do the following programs print?

```
void foo(  int n  )
{
      int data;
      if ( n == 0 ) return;
      scanf("%d", &data);
      foo ( n – 1 );
      printf("%d\n", data);
}
main ( )
{     int k = 5;
      foo ( k );
}
```

```
void foo(  int n  )
{
      int data;
      if ( n == 0 ) return;
      foo ( n – 1 );
      scanf("%d", &data);
      printf("%d\n", data);
}
main ( )
{     int k = 5;
      foo ( k );
}
```

```
void foo(  int n  )
{
      int data;
      if ( n == 0 ) return;
      scanf("%d", &data);
      printf("%d\n", data);
      foo ( n – 1 );
}
main ( )
{     int k = 5;
      foo ( k );
}
```

Printing cumulative sum -- *will this work?*

```
int foo(  int n  )
{
      int data, sum ;
      if ( n == 0 ) return 0;
      scanf("%d", &data);
      sum = data + foo ( n – 1 );
      printf("%d\n", sum);
      return sum;
}
main ( ) {
      int k = 5;
      foo ( k );
}
```

Input: 1 2 3   4 5

Output: 5 9 12 14 15

How to rewrite this so that the output is: 1 3 6 10 15 ?

Printing cumulative sum (two ways)

```
int foo(  int n  )
{
      int data, sum ;
      if ( n == 0 ) return 0;
      sum = foo ( n – 1 );
      scanf("%d", &data);
      sum = sum + data;
      printf("%d\n", sum);
      return sum;
}
main ( ) {
      int k = 5;
      foo ( k );
}
```

Input: 1 2 3   4   5

Output: 1 3 6 10 15

```
void foo(  int n, int sum  )
{
      int data ;
      if ( n == 0 ) return 0;
      scanf("%d", &data);
      sum = sum + data;
      printf("%d\n", sum);
      foo( k – 1, sum ) ;
}
main ( ) {
      int k = 5;
      foo ( k, 0 );
}
```

Paying with fewest coins

- A country has coins of denomination 3, 5 and 10, respectively.
- We are to write a function canchange(k) that returns –1 if it is not possible to pay a value of k using these coins.
    - Otherwise it returns the minimum number of coins needed to make the payment.

- For example, canchange(7) will return –1.
- On the other hand, canchange(14) will return 4 because 14 can be paid as 3+3+3+5 and there is no other way to pay with fewer coins.
- Finally, 15 can be changed as 3+3+3+3+3, 5+5+5, 5+10, so canchange(15) will return 2.

Paying with fewest coins

```
int canchange( int k )
{
        int a;
        if (k==0) return 0;
        if ( ____________ ) return 1;
        if (k < 3)  _________________ ;

        a = canchange( ____________ ); if (a > 0) return ____________ ;
        a = canchange(k – 5); if (a > 0) return _________________ ;
        a = canchange( ____________ ); if (a > 0) return _____________ ;
        return –1;
}
```

Paying with fewest coins

```
int canchange( int k )
{
        int a;
        if (k==0) return 0;
        if ( (k ==3) || (k == 5) || (k == 10) ) return 1;
        if (k < 3)  return –1  ;

        a = canchange( k – 10 ); if (a > 0) return a+1 ;
        a = canchange( k – 5 ); if (a > 0) return a+1 ;
        a = canchange( k – 3 ); if (a > 0) return a+1 ;
        return –1;
}
```

Exercise: Rewrite this code if the denominations are 3, 8, and 10. Do you see a problem? Repair it.

Practice Problems

1. Write a recursive function to search for an element in an array
2. Write a recursive function to count the digits of a positive integer (do also for sum of digits)
3. Write a recursive function to reverse a null-terminated string
4. Write a recursive function to convert a decimal number to binary
5. Write a recursive function to check if a string is a palindrome or not
6. Write a recursive function to copy one array to another

Note:

- For each of the above, write the main functions to call the recursive function also
- Practice problems are just for practicing recursion, recursion is not necessarily the most efficient way of doing them

Advanced topic (optional)

How are recursive calls implemented?

What we have seen ….

- **Activation record gets pushed into the stack when a function call is made.**
- **Activation record is popped off the stack when the function returns.**

In recursion, a function calls itself.

- **Several function calls going on, with none of the function calls returning back.**
    - **Activation records are pushed onto the stack continuously.**
    - **Large stack space required.**

- Activation records keep popping off, when the termination condition of recursion is reached.

We shall illustrate the process by an example of computing factorial.

- Activation record looks like:

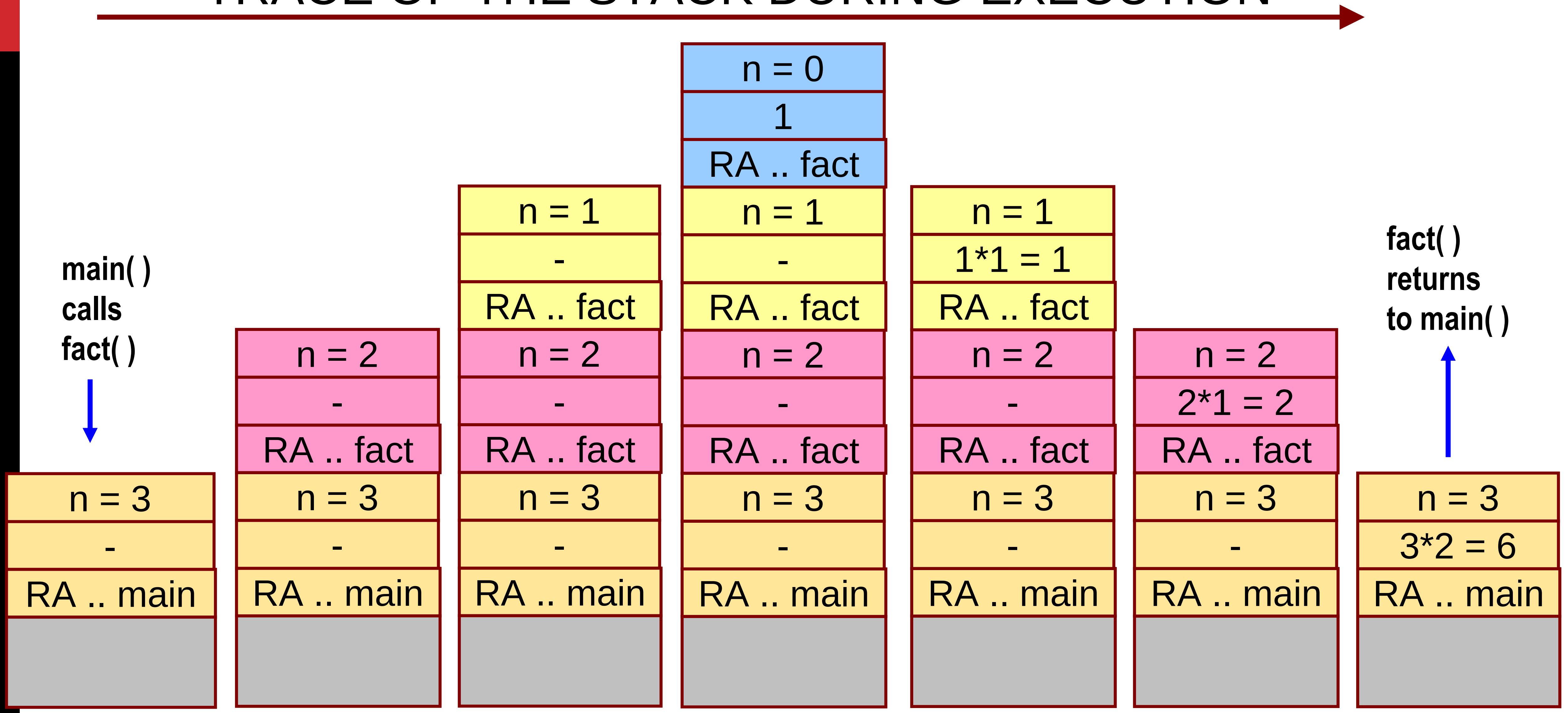
Actual Parameters
Local Variables
Return Value
Return Address
. . .

Example:: main( ) calls fact(3)

```
main()
{
  int  n;
  n = 3;
  printf (“%d \n”, fact(n) );
}
```

```
int  fact (n)
int  n;
{
         if   (n = = 0)
                  return (1);
         else
                  return  (n * fact(n-1));
}
```

TRACE OF THE STACK DURING EXECUTION

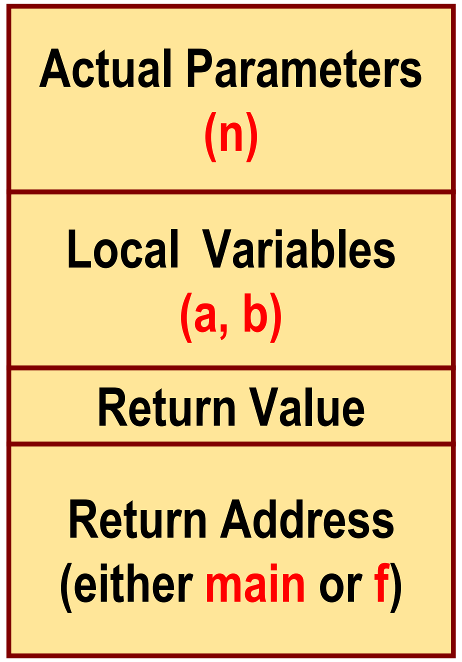


Do Yourself

Trace the activation records for the following version of Fibonacci sequence.

```
          #include <stdio.h>
          int   f (int n)
          {
               int a, b;
               if  (n  < 2)   return (n);
               else  {
X    →          a = f(n-1);
Y    →          b = f(n-2);
                return (a+b);  }
          }

          main( ) {
              printf("Fib(4) is: %d \n", f(4));
          }
```


Actual Parameters
(n)
Local Variables
(a, b)
Return Value
Return Address
(either main or f)