Number Representation

Part I

Basics of Number System

- We are accustomed to using the so-called decimal number system
  - Ten digits :: 0,1,2,3,4,5,6,7,8,9
  - Every digit position has a weight which is a power of 10
  - Base or radix is 10

Example:

$234 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$

$250.67 = 2 \times 10^2 + 5 \times 10^1 + 0 \times 10^0 + 6 \times 10^{-1} + 7 \times 10^{-2}$

Binary Number System

- Two digits:
  - 0 and 1
  - Every digit position has a weight which is a power of 2
  - Base or radix is 2
- Example:

$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$

$101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$

Positional Number Systems (General)

Decimal Numbers:

- **10 Symbols {0,1,2,3,4,5,6,7,8,9}, Base or Radix is 10**
- **$136.25 = 1 \times 10^2 + 3 \times 10^1 + 6 \times 10^0 + 2 \times 10^{-1} + 3 \times 10^{-2}$**

Positional Number Systems (General)

Decimal Numbers:

- **10 Symbols {0,1,2,3,4,5,6,7,8,9}, Base or Radix is 10**
- **$136.25 = 1 \times 10^{2} + 3 \times 10^{1} + 6 \times 10^{0} + 2 \times 10^{-1} + 3 \times 10^{-2}$**

Binary Numbers:

- **2 Symbols {0,1}, Base or Radix is 2**
- **$101.01 = 1 \times 2^{2} + 0 \times 2^{1} + 1 \times 2^{0} + 0 \times 2^{-1} + 1 \times 2^{-2}$**

Positional Number Systems (General)

Decimal Numbers:

- **10 Symbols {0,1,2,3,4,5,6,7,8,9}, Base or Radix is 10**
- **$136.25 = 1 \times 10^2 + 3 \times 10^1 + 6 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2}$**

Binary Numbers:

- **2 Symbols {0,1}, Base or Radix is 2**
- **$101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$**

Octal Numbers:

- **8 Symbols {0,1,2,3,4,5,6,7}, Base or Radix is 8**
- **$621.03 = 6 \times 8^2 + 2 \times 8^1 + 1 \times 8^0 + 0 \times 8^{-1} + 3 \times 8^{-2}$**

Positional Number Systems (General)

Decimal Numbers:

- **10 Symbols {0,1,2,3,4,5,6,7,8,9}, Base or Radix is 10**
- **136.25 = $1 \times 10^2 + 3 \times 10^1 + 6 \times 10^0 + 2 \times 10^{-1} + 3 \times 10^{-2}$**

Binary Numbers:

- **2 Symbols {0,1}, Base or Radix is 2**
- **101.01 = $1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$**

Octal Numbers:

- **8 Symbols {0,1,2,3,4,5,6,7}, Base or Radix is 8**
- **621.03 = $6 \times 8^2 + 2 \times 8^1 + 1 \times 8^0 + 0 \times 8^{-1} + 3 \times 8^{-2}$**

Hexadecimal Numbers:

- **16 Symbols {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}, Base is 16**
- **6AF.3C = $6 \times 16^2 + 10 \times 16^1 + 15 \times 16^0 + 3 \times 16^{-1} + 12 \times 16^{-2}$**

Binary-to-Decimal Conversion

- Each digit position of a binary number has a weight
  - Some power of 2
- A binary number:

$$B = b_{n-1}\, b_{n-2} \ldots\ldots b_1\, b_0\, .\, b_{-1}\, b_{-2} \ldots\ldots b_{-m}$$

Corresponding value in decimal:

$$D = \sum_{i=-m}^{n-1} b_i\, 2^i$$

Examples

101011 ➔ $1\times2^5 + 0\times2^4 + 1\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0$

$= 43$

$(101011)_2 = (43)_{10}$

.0101 ➔ $0\times2^{-1} + 1\times2^{-2} + 0\times2^{-3} + 1\times2^{-4}$

$= .3125$

$(.0101)_2 = (.3125)_{10}$

101.11 ➔ $1\times2^2 + 0\times2^1 + 1\times2^0 + 1\times2^{-1} + 1\times2^{-2}$

$= 5.75$

$(101.11)_2 = (5.75)_{10}$

Decimal to Binary: Integer Part

- Consider the integer and fractional parts separately.
- For the integer part:
  - Repeatedly divide the given number by 2, and go on accumulating the remainders, until the number becomes zero.
  - Arrange the remainders in reverse order.

Base	Numb	Rem
2	89	
2	44	1
2	22	0
2	11	0
2	5	1
2	2	1
2	1	0
	0	1

$(89)_{10} = (1011001)_2$

Decimal to Binary: Integer Part

- Consider the integer and fractional parts separately.
- For the integer part:
  - Repeatedly divide the given number by 2, and go on accumulating the remainders, until the number becomes zero.
  - Arrange the remainders in reverse order.

Base	Numb	Rem
2	89	
2	44	1
2	22	0
2	11	0
2	5	1
2	2	1
2	1	0
	0	1

$(89)_{10} = (1011001)_2$

2	66	
2	33	0
2	16	1
2	8	0
2	4	0
2	2	0
2	1	0
	0	1

$(66)_{10} = (1000010)_2$

Decimal to Binary: Integer Part

- Consider the integer and fractional parts separately.
- For the integer part:
  - Repeatedly divide the given number by 2, and go on accumulating the remainders, until the number becomes zero.
  - Arrange the remainders in reverse order.

Base	Numb	Rem
2	89	
2	44	1
2	22	0
2	11	0
2	5	1
2	2	1
2	1	0
	0	1

$(89)_{10} = (1011001)_2$

2	66	
2	33	0
2	16	1
2	8	0
2	4	0
2	2	0
2	1	0
	0	1

$(66)_{10} = (1000010)_2$

2	239	
2	119	1
2	59	1
2	29	1
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

$(239)_{10} = (11101111)_2$

Decimal to Binary: Fraction Part

- Repeatedly multiply the given fraction by 2.
  - Accumulate the integer part (0 or 1).
  - If the integer part is 1, chop it off.
- Arrange the integer parts in the order they are obtained.

Example: 0.634

$.634 \times 2 = 1.268$

$.268 \times 2 = 0.536$

$.536 \times 2 = 1.072$

$.072 \times 2 = 0.144$

$.144 \times 2 = 0.288$

:

:

$(.634)_{10} = (.10100......)_2$

Decimal to Binary: Fraction Part

- Repeatedly multiply the given fraction by 2.
  - Accumulate the integer part (0 or 1).
  - If the integer part is 1, chop it off.
- Arrange the integer parts in the order they are obtained.

Example: 0.634

$.634 \times 2 = 1.268$

$.268 \times 2 = 0.536$

$.536 \times 2 = 1.072$

$.072 \times 2 = 0.144$

$.144 \times 2 = 0.288$

:

:

$(.634)_{10} = (.10100......)_2$

Example: 0.0625

$.0625 \times 2 = 0.125$

$.1250 \times 2 = 0.250$

$.2500 \times 2 = 0.500$

$.5000 \times 2 = 1.000$

$(.0625)_{10} = (.0001)_2$

Decimal to Binary: Fraction Part

- Repeatedly multiply the given fraction by 2.
  - Accumulate the integer part (0 or 1).
  - If the integer part is 1, chop it off.
- Arrange the integer parts in the order they are obtained.

Example: 0.634

$.634 \times 2 = 1.268$

$.268 \times 2 = 0.536$

$.536 \times 2 = 1.072$

$.072 \times 2 = 0.144$

$.144 \times 2 = 0.288$

:

:

$(.634)_{10} = (.10100\ldots\ldots)_2$

Example: 0.0625

$.0625 \times 2 = 0.125$

$.1250 \times 2 = 0.250$

$.2500 \times 2 = 0.500$

$.5000 \times 2 = 1.000$

$(.0625)_{10} = (.0001)_2$

$(37)_{10} = (100101)_2$

$(.0625)_{10} = (.0001)_2$

$(37.0625)_{10} = (100101.0001)_2$

Hexadecimal Number System

- A compact way of representing binary numbers
- 16 different symbols (radix = 16)

0	→	0000	8	→	1000
1	→	0001	9	→	1001
2	→	0010	A	→	1010
3	→	0011	B	→	1011
4	→	0100	C	→	1100
5	→	0101	D	→	1101
6	→	0110	E	→	1110
7	→	0111	F	→	1111

Binary-to-Hexadecimal Conversion

- For the integer part,
  - Scan the binary number from right to left
  - Translate each group of four bits into the corresponding hexadecimal digit
    - Add leading zeros if necessary

- For the fractional part,
  - Scan the binary number from left to right
  - Translate each group of four bits into the corresponding hexadecimal digit
    - Add trailing zeros if necessary

Example

1. $(\underline{1011}\ \underline{0100}\ \underline{0011})_2 = (B43)_{16}$
2. $(\underline{10}\ \underline{1010}\ \underline{0001})_2 = (2A1)_{16}$
3. $(.\underline{1000}\ \underline{010})_2 = (.84)_{16}$
4. $(\underline{101}\ .\ \underline{0101}\ \underline{111})_2 = (5.5E)_{16}$

Hexadecimal-to-Binary Conversion

- Translate every hexadecimal digit into its 4-bit binary equivalent

- Examples:

  $(3A5)_{16} = (0011\ 1010\ 0101)_2$

  $(12.3D)_{16} = (0001\ 0010\ .\ 0011\ 1101)_2$

  $(1.8)_{16} = (0001\ .\ 1000)_2$

Number Representation

Part II

Unsigned Binary Numbers

- An n-bit binary number

  $B = b_{n-1}b_{n-2} \ldots. b_2b_1b_0$

  - 2^n distinct combinations are possible, 0 to 2^n-1.
- For example, for n = 3, there are 8 distinct combinations
  - 000, 001, 010, 011, 100, 101, 110, 111
- Range of numbers that can be represented

  n=8 ➔ 0 to 2^8-1 (255)

  n=16 ➔ 0 to 2^{16}-1 (65535)

  n=32 ➔ 0 to 2^{32}-1 (4294967295)

Signed Integer Representation

- Many of the numerical data items that are used in a program are signed (positive or negative)
  - Question:: How to represent sign?

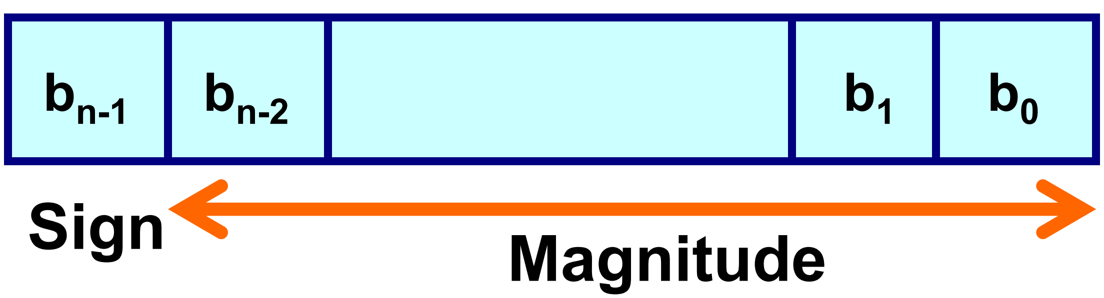
- Three possible approaches:
  - Sign-magnitude representation
  - One’s complement representation
  - Two’s complement representation

Sign-magnitude Representation

- For an n-bit number representation
  - The most significant bit (MSB) indicates sign

    0 → positive

    1 → negative
  - The remaining n-1 bits represent magnitude



Contd.

- Range of numbers that can be represented:

  Maximum :: $+ (2^{n-1} - 1)$

  Minimum :: $- (2^{n-1} - 1)$

- A problem:

  Two different representations of zero

  +0 → 0 000….0

  -0 → 1 000….0

One's Complement Representation

- Basic idea:
  - Positive numbers are represented exactly as in sign-magnitude form
  - Negative numbers are represented in 1's complement form
- How to compute the 1's complement of a number?
  - Complement every bit of the number (1→0 and 0→1)
  - MSB will indicate the sign of the number

    0 → positive

    1 → negative

Example: n=4

0000 → +0	1000 → -7
0001 → +1	1001 → -6
0010 → +2	1010 → -5
0011 → +3	1011 → -4
0100 → +4	1100 → -3
0101 → +5	1101 → -2
0110 → +6	1110 → -1
0111 → +7	1111 → -0

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 1's complement of 0100 = 1011

Contd.

- Range of numbers that can be represented:

  Maximum :: $+ (2^{n-1} - 1)$

  Minimum :: $- (2^{n-1} - 1)$

- A problem:

  Two different representations of zero.

  +0 → 0 000….0

  -0 → 1 111….1

- Advantage of 1's complement representation
  - Subtraction can be done using addition
  - Leads to substantial saving in circuitry

Two's Complement Representation

- Basic idea:
  - Positive numbers are represented exactly as in sign-magnitude form
  - Negative numbers are represented in 2's complement form
- How to compute the 2's complement of a number?
  - Complement every bit of the number (1→0 and 0→1), and then add one to the resulting number
  - MSB will indicate the sign of the number

    0 → positive

    1 → negative

Example: n=4

0000 → +0		1000 → -8	
0001 → +1		1001 → -7	
0010 → +2		1010 → -6	
0011 → +3		1011 → -5	
0100 → +4		1100 → -4	
0101 → +5		1101 → -3	
0110 → +6		1110 → -2	
0111 → +7		1111 → -1	

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 2's complement of 0100 = 1011+1 = 1100

Rule : Value = – msb*2$^{(n-1)}$ + [unsigned value of rest]

Example: 0110 = 0 + 6 = 6 1110 = – 8 + 6 = – 2

Contd.

- Range of numbers that can be represented:

  Maximum :: $+ (2^{n-1} - 1)$

  Minimum :: $- 2^{n-1}$
- Advantage:
  - Unique representation of zero
  - Subtraction can be done using addition
  - Leads to substantial saving in circuitry
- Almost all computers today use the 2's complement representation for storing negative numbers

Adding Binary Numbers

- Basic Rules:
  - 0+0=0
  - 0+1=1
  - 1+0=1
  - 1+1=0 (carry 1)

- Example:

```
01101001
00110100
-------------
10011101
```

Subtraction Using Addition: 1's Complement

- How to compute A – B ?
  - Compute the 1's complement of B (say, B_1).
  - Compute $R = A + B_1$
  - If the carry obtained after addition is '1'
    - Add the carry back to R (called *end-around carry*)
    - That is, R = R + 1
    - The result is a positive number

    Else
    - The result is negative, and is in 1's complement form

Example 1: 6 – 2

1's complement of 2 = 1101

6	::	0110	**A**
-2	::	1101	**B_1**
		1 0011	**R**
		1	
		0100	➔ +4

End-around carry

Assume 4-bit representations

Since there is a carry, it is added back to the result

The result is positive

Example 2: 3 – 5

1's complement of 5 = 1010

3	::	0011	**A**
-5	::	1010	**B_1**
		1101	**R**

↓

-2

Assume 4-bit representations

Since there is no carry, the result is negative

1101 is the 1's complement of 0010, that is, it represents –2

Subtraction Using Addition: 2's Complement

- How to compute A – B ?
  - Compute the 2's complement of B (say, B_2)
  - Compute $R = A + B_2$
  - If the carry obtained after addition is '1'
    - Ignore the carry
    - The result is a positive number

    Else
    - The result is negative, and is in 2's complement form

Example 1: 6 – 2

2's complement of 2 = 1101 + 1 = 1110

6	::	0110	**A**
-2	::	1110	**B_2**
		1 ~~0100~~	**R**

Ignore carry

+4

Assume 4-bit representations

Presence of carry indicates that the result is positive

No need to add the end-around carry like in 1's complement

Example 2: 3 – 5

2's complement of 5 = 1010 + 1 = 1011

3	::	0011	**A**
-5	::	1011	$\mathbf{B_2}$
		1110	**R**

⬇

-2

Assume 4-bit representations

Since there is no carry, the result is negative

1110 is the 2's complement of 0010, that is, it represents –2

2's complement arithmetic: More Examples

- Example 1: 18-11 = ?
- 18 is represented as 00010010
- 11 is represented as 00001011
  - 1's complement of 11 is 11110100
  - 2's complement of 11 is 11110101
- Add 18 to 2's complement of 11

```
   00010010
+  11110101
---------------
   00000111 (with a carry of 1
                which is ignored)
```

00000111 is 7

- Example 2: 7 - 9 = ?
- 7 is represented as 00000111
- 9 is represented as 00001001
  - 1's complement of 9 is 11110110
  - 2's complement of 9 is 11110111
  - Add 7 to 2's complement of 9

```
   00000111
+  11110111
---------------
   11111110 (with a carry of 0
                which is ignored)
```

11111110 is -2

Number Representation

Part III

Overflow/Underflow:

Adding two +ve (-ve) numbers should not produce a –ve (+ve) number. If it does, overflow (underflow) occurs

Overflow/Underflow:

Adding two +ve (-ve) numbers should not produce a –ve (+ve) number. If it does, overflow (underflow) occurs

Another equivalent condition : carry in and carry out from Most Significant Bit (MSB) differ.

Overflow/Underflow:

Adding two +ve (-ve) numbers should not produce a –ve (+ve) number. If it does, overflow (underflow) occurs

Another equivalent condition : carry in and carry out from Most Significant Bit (MSB) differ.

```
(64)  01000000
( 4)  00000100
      --------------
 (68) 01000100
```

```
carry (out)(in)
         0    0
```

Overflow/Underflow:

Adding two +ve (-ve) numbers should not produce a –ve (+ve) number. If it does, overflow (underflow) occurs

Another equivalent condition : carry in and carry out from Most Significant Bit (MSB) differ.

```
(64)  01000000
( 4)  00000100
      --------------
(68)  01000100
```

carry (out)(in)
0 0

```
(64)  01000000
(96)  01100000
      --------------
(-96) 10100000
```

carry out in
0 1

differ:

overflow

Floating-point Numbers

- The representations discussed so far applies only to integers
  - Cannot represent numbers with fractional parts
- We can assume a decimal point before a signed number
  - In that case, pure fractions (without integer parts) can be represented
- We can also assume the decimal point somewhere in between
  - This lacks flexibility
  - Very large and very small numbers cannot be represented

Representation of Floating-Point Numbers

- A floating-point number F is represented by a doublet <M,E> :

  $F = M \times B^E$

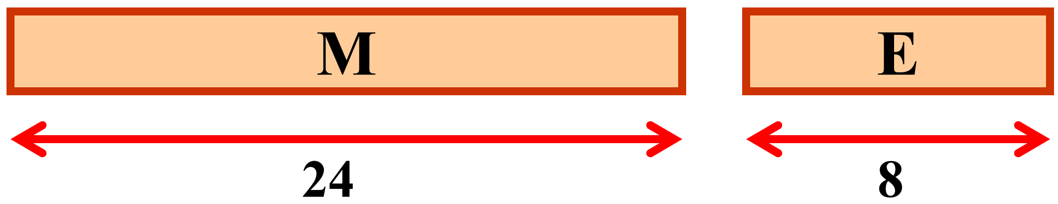
  - B → exponent base (usually 2)
  - M → mantissa
  - E → exponent

  - M is usually represented in 2's complement form, with an implied binary point before it
- For example,

  In decimal, 0.235×10^6

  In binary, 0.101011×2^{0110}

Example :: 32-bit representation



- M represents a 2's complement fraction

  1 > M > -1
- E represents the exponent (in 2's complement form)

  127 > E > -128

- Points to note:
  - The number of significant digits depends on the number of bits in M
    - 6 significant digits for 24-bit mantissa
  - The range of the number depends on the number of bits in E
    - 10^{38} to 10^{-38} for 8-bit exponent.

- Sign bit is added in front to represent both +ve and –ve numbers
- The representation shown for floating-point numbers as shown is just for illustration
- The actual representation is a little more complex, we will not do here
  - Example: IEEE 754 Floating Point format