Number Systems

Number Representation

Topics to be Discussed

- How are numeric data items actually stored in computer memory?
- How much space (memory locations) is allocated for each type of data?
  - int, float, char, etc.
- How are characters and strings stored in memory?

Number System :: The Basics

- We are accustomed to using the so-called *decimal number system*.
  - Ten digits :: 0,1,2,3,4,5,6,7,8,9
  - Every digit position has a weight which is a power of 10.
  - Base or radix is 10.
- Example:

$234 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$

$250.67 = 2 \times 10^2 + 5 \times 10^1 + 0 \times 10^0 + 6 \times 10^{-1} + 7 \times 10^{-2}$

Binary Number System

- Two digits:
  - 0 and 1.
  - Every digit position has a weight which is a power of 2.
  - *Base* or *radix* is 2.
- Example:

$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$

$101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$

Counting with Binary Numbers

0
1
10
11
100
101
110
111
1000
.

Multiplication and Division with base

- Multiplication with 10 (decimal system)

  435 x 10 = 4350

- Multiplication with 10 (=2) (binary system)

  1101 x 10 = 11010

- Division by 10 (decimal system)

  435 / 10 = 43.5

- Division by 10 (=2) (binary system)

  1101 / 10 = 110.1

Left Shift and add zero at right end

Right shift and drop right most digit or shift after decimal point

Adding two bits

0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = 10

carry

Carries

	1	1	1	0	
		1	0	1	1
+		1	1	1	0
	1	1	0	0	1

Binary addition: Another example

The initial carry in is implicitly 0

	1	1	0	0		(Carries)
		1	1	0	1	
+		1	1	0	0	
	1	1	0	0	1	(Sum)

most significant bit (MSB)

least significant bit (LSB)

Binary-to-Decimal Conversion

- Each digit position of a binary number has a weight.
  - Some power of 2.
- A binary number:

$$B = b_{n-1}\ b_{n-2}\ .....b_1\ b_0\ .\ b_{-1}\ b_{-2}\ .....\ b_{-m}$$

Corresponding value in decimal:

$$D = \sum_{i=-m}^{n-1} b_i\, 2^i$$

Examples

1. 101011 ➔ $1x2^5 + 0x2^4 + 1x2^3 + 0x2^2 + 1x2^1 + 1x2^0$

   $= 43$

   $(101011)_2 = (43)_{10}$

2. .0101 ➔ $0x2^{-1} + 1x2^{-2} + 0x2^{-3} + 1x2^{-4}$

   $= .3125$

   $(.0101)_2 = (.3125)_{10}$

3. 101.11 ➔ $1x2^2 + 0x2^1 + 1x2^0 + 1x2^{-1} + 1x2^{-2}$

   5.75

   $(101.11)_2 = (5.75)_{10}$

Decimal-to-Binary Conversion

- Consider the integer and fractional parts separately.
- For the integer part,
  - Repeatedly divide the given number by 2, and go on accumulating the remainders, until the number becomes zero.
  - Arrange the remainders *in reverse order*.
- For the fractional part,
  - Repeatedly multiply the given fraction by 2.
    - Accumulate the integer part (0 or 1).
    - If the integer part is 1, chop it off.
  - Arrange the integer parts *in the order* they are obtained.

Example 1 :: 239

2	239	
2	119	--- 1
2	59	--- 1
2	29	--- 1
2	14	--- 1
2	7	--- 0
2	3	--- 1
2	1	--- 1
2	0	--- 1

$(239)_{10} = (11101111)_2$

Example 2 :: 64

2	64	
2	32	--- 0
2	16	--- 0
2	8	--- 0
2	4	--- 0
2	2	--- 0
2	1	--- 0
2	0	--- 1

$(64)_{10} = (1000000)_2$

Example 3 :: .634

.634 x 2 = **1**.268

.268 x 2 = **0**.536

.536 x 2 = **1**.072

.072 x 2 = **0**.144

.144 x 2 = **0**.288

:

:

$(.634)_{10} = (.10100......)_2$

Example 4 :: 37.0625

$(37)_{10} = (100101)_2$

$(.0625)_{10} = (.0001)_2$

$\therefore (37.0625)_{10} = (100101\,.\,0001)_2$

Hexadecimal Number System

- A compact way of representing binary numbers.
- 16 different symbols (radix = 16).

0	→	0000	8	→	1000
1	→	0001	9	→	1001
2	→	0010	A	→	1010
3	→	0011	B	→	1011
4	→	0100	C	→	1100
5	→	0101	D	→	1101
6	→	0110	E	→	1110
7	→	0111	F	→	1111

Binary-to-Hexadecimal Conversion

- For the integer part,
  - Scan the binary number from *right to left*.
  - Translate each group of four bits into the corresponding hexadecimal digit.
    - Add *leading* zeros if necessary.
- For the fractional part,
  - Scan the binary number from *left to right*.
  - Translate each group of four bits into the corresponding hexadecimal digit.
    - Add *trailing* zeros if necessary.

Example

1. $(\underline{1011}\ \underline{0100}\ \underline{0011})_2 = (B43)_{16}$
2. $(\underline{10}\ \underline{1010}\ \underline{0001})_2 = (2A1)_{16}$
3. $(.\underline{1000}\ \underline{010})_2 = (.84)_{16}$
4. $(\underline{101}\ .\ \underline{0101}\ \underline{111})_2 = (5.5E)_{16}$

Hexadecimal-to-Binary Conversion

- Translate every hexadecimal digit into its 4-bit binary equivalent.
- Examples:

$(3A5)_{16} = (0011\ 1010\ 0101)_2$

$(12.3D)_{16} = (0001\ 0010\ .\ 0011\ 1101)_2$

$(1.8)_{16} = (0001\ .\ 1000)_2$

Unsigned Binary Numbers

- An n-bit binary number

  $B = b_{n-1}b_{n-2} \ldots. b_2b_1b_0$

  - 2^n distinct combinations are possible, 0 to 2^n-1.
- For example, for n = 3, there are 8 distinct combinations.
  – 000, 001, 010, 011, 100, 101, 110, 111
- Range of numbers that can be represented

  n=8 → 0 to 2^8-1 (255)

  n=16 → 0 to 2^{16}-1 (65535)

  n=32 → 0 to 2^{32}-1 (4294967295)

Signed Integer Representation

- Many of the numerical data items that are used in a program are signed (positive or negative).
  - Question:: How to represent sign?
- Three possible approaches:
  - Sign-magnitude representation
  - One's complement representation
  - Two's complement representation

Sign-magnitude Representation

- For an n-bit number representation
  - The most significant bit (MSB) indicates sign
    - 0 → positive
    - 1 → negative
  - The remaining n-1 bits represent magnitude.

Contd.

- Range of numbers that can be represented:

  Maximum :: $+ (2^{n-1} - 1)$

  Minimum :: $- (2^{n-1} - 1)$

- A problem:

  Two different representations of zero.

  +0 → 0 000….0

  -0 → 1 000….0

One's Complement Representation

- Basic idea:
  - Positive numbers are represented exactly as in sign-magnitude form.
  - Negative numbers are represented in 1's complement form.
- How to compute the 1's complement of a number?
  - Complement every bit of the number (1→0 and 0→1).
  - MSB will indicate the sign of the number.

    0 → positive

    1 → negative

Example :: n=4

0000 → +0	1000 → -7
0001 → +1	1001 → -6
0010 → +2	1010 → -5
0011 → +3	1011 → -4
0100 → +4	1100 → -3
0101 → +5	1101 → -2
0110 → +6	1110 → -1
0111 → +7	1111 → -0

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 1's complement of 0100 = 1011

Contd.

- Range of numbers that can be represented:

  Maximum :: $+ (2^{n-1} - 1)$

  Minimum :: $- (2^{n-1} - 1)$

- A problem:

  Two different representations of zero.

  +0 → 0 000….0

  -0 → 1 111….1

- Advantage of 1's complement representation
  - Subtraction can be done using addition.
  - Leads to substantial saving in circuitry.

Two's Complement Representation

- Basic idea:
  - Positive numbers are represented exactly as in sign-magnitude form.
  - Negative numbers are represented in 2's complement form.
- How to compute the 2's complement of a number?
  - Complement every bit of the number (1→0 and 0→1), and then *add one* to the resulting number.
  - MSB will indicate the sign of the number.

    0 → positive

    1 → negative

Example :: n=4

0000 → +0		1000 → -8	
0001 → +1		1001 → -7	
0010 → +2		1010 → -6	
0011 → +3		1011 → -5	
0100 → +4		1100 → -4	
0101 → +5		1101 → -3	
0110 → +6		1110 → -2	
0111 → +7		1111 → -1	

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 2's complement of 0100 = 1011+1 = 1100

Contd.

- In C
  - short int
    - 16 bits ➔ $+(2^{15}-1)$ to -2^{15}
  - int
    - 32 bits ➔ $+(2^{31}-1)$ to -2^{31}
  - long int
    - 64 bits ➔ $+(2^{63}-1)$ to -2^{63}

Contd.

- Range of numbers that can be represented:

  Maximum :: $+(2^{n-1} - 1)$

  Minimum :: -2^{n-1}

- Advantage:
  - *Unique representation of zero*.
  - Subtraction can be done using addition.
  - Leads to substantial saving in circuitry.
- Almost all computers today use the 2's complement representation for storing negative numbers.

Subtraction Using Addition :: 1's Complement

- How to compute A – B ?
  - Compute the 1's complement of B (say, B_1).
  - Compute $R = A + B_1$
  - If the carry obtained after addition is '1'
    - Add the carry back to R (called *end-around carry*).
    - That is, R = R + 1.
    - The result is a positive number.

    Else
    - The result is negative, and is in 1's complement form.

Example 1 :: 6 – 2

1's complement of 2 = 1101

			A
6	::	0110	B_1
-2	::	1101	R
		1 0011	
		1	
		0100	→ +4

End-around carry

Assume 4-bit representations.

Since there is a carry, it is added back to the result.

The result is positive.

Example 2 :: 3 – 5

1's complement of 5 = 1010

3	::	0011	**A**
-5	::	1010	**B_1**
		1101	**R**
		-2	

Assume 4-bit representations.

Since there is no carry, the result is negative.

1101 is the 1's complement of 0010, that is, it represents –2.

Subtraction Using Addition :: 2's Complement

- How to compute A – B ?
  - Compute the 2's complement of B (say, B_2).
  - Compute $R = A + B_2$
  - Ignore carry if it is there.
  - The result is in 2's complement form.

Example 1 :: 6 – 2

2's complement of 2 = 1101 + 1 = 1110

6	::	0110	A
-2	::	1110	B_2
		1 0100	R

Ignore carry

+4

Example 2 :: 3 – 5

2's complement of 5 = 1010 + 1 = 1011

			A
3	::	0011	**B_2**
-5	::	1011	**R**
		1110	
		-2	

Example 3 :: -3 – 5

2's complement of 3 = 1100 + 1 = 1101

2's complement of 5 = 1010 + 1 = 1011

-3 :: 1101

-5 :: 1011

1 1000

Ignore carry

-8

Floating-point Numbers

- The representations discussed so far applies only to integers.
  - Cannot represent numbers with fractional parts.
- We can assume a decimal point before a 2's complement number.
  - In that case, pure fractions (without integer parts) can be represented.
- We can also assume the decimal point somewhere in between.
  - This lacks flexibility.
  - Very large and very small numbers cannot be represented.

Representation of Floating-Point Numbers

- A floating-point number F is represented by a doublet <M,E> :

  $F = M \times B^E$

  - B → exponent base (usually 2)
  - M → mantissa
  - E → exponent

  – M is usually represented in 2's complement form, with an implied decimal point before it.

- For example,

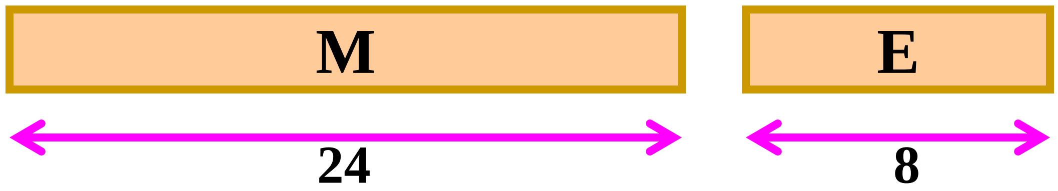
  In decimal,

  0.235×10^6

  In binary,

  0.101011×2^{0110}

Example :: 32-bit representation



- M represents a 2's complement fraction

  $1 > M > -1$

- E represents the exponent (in 2's complement form)

  $127 > E > -128$

- Points to note:
  - The number of *significant digits* depends on the number of bits in M.
    - 6 significant digits for 24-bit mantissa.
  - The *range* of the number depends on the number of bits in E.
    - 10^{38} to 10^{-38} for 8-bit exponent.

A Warning

- The representation for floating-point numbers as shown is just for illustration.
- The actual representation is a little more complex.
- In C:
  - float :: 32-bit representation
  - double :: 64-bit representation

Representation of Characters

- Many applications have to deal with non-numerical data.
  - Characters and strings.
  - There must be a standard mechanism to represent alphanumeric and other characters in memory.
- Three standards in use:
  - Extended Binary Coded Decimal Interchange Code (EBCDIC)
    - Used in older IBM machines.
  - American Standard Code for Information Interchange (ASCII)
    - Most widely used today.
  - UNICODE
    - Used to represent all international characters.
    - Used by Java.

ASCII Code

- Each individual character is numerically encoded into a unique 7-bit binary code.
  - A total of 2^7 or 128 different characters.
  - A character is normally encoded in a byte (8 bits), with the MSB not been used.
- The binary encoding of the characters follow a regular ordering.
  - Digits are ordered consecutively in their proper numerical sequence (0 to 9).
  - Letters (uppercase and lowercase) are arranged consecutively in their proper alphabetic order.

Some Common ASCII Codes

‘A’ :: 41 (H) 65 (D)

‘B’ :: 42 (H) 66 (D)

…………

‘Z’ :: 5A (H) 90 (D)

‘a’ :: 61 (H) 97 (D)

‘b’ :: 62 (H) 98 (D)

…………

‘z’ :: 7A (H) 122 (D)

‘0’ :: 30 (H) 48 (D)

‘1’ :: 31 (H) 49 (D)

…………

‘9’ :: 39 (H) 57 (D)

‘(‘ :: 28 (H) 40 (D)

‘+’ :: 2B (H) 43 (D)

‘?’ :: 3F (H) 63 (D)

‘\n’ :: 0A (H) 10 (D)

‘\0’ :: 00 (H) 00 (D)

Character Strings

- Two ways of representing a sequence of characters in memory.
  - The first location contains the number of characters in the string, followed by the actual characters.

5	H	e	l	l	o

  - The characters follow one another, and is terminated by a special delimiter.

H	e	l	l	o	⊥

String Representation in C

- In C, the second approach is used.
  - The '\0' character is used as the string delimiter.
- Example:

  "Hello" ➔

H	**e**	**l**	**l**	**o**	**'\0'**

- A null string "" occupies one byte in memory.
  - Only the '\0' character.