Advanced graph theory: Tutorial 3: CS60047 Autumn September 16, 2022

1. If the bipartite graph $G_{m,n}$ has no $K_{r.s}$, as discussed in class (see class slides), argue from first principles the correctness of the inequality $\sum_{x \in V_2} \binom{d(x)}{r} \leq (s-1)\binom{m}{r}$ based on two different ways of counting the r-tuple W and vertex $x \in V_2$ pairings.

2. Write the contrapositive of the implication statement in Question 1 above. Show that it holds for the graph $G_{m,n} = K_{3,4}$ for $r = 3$ and $s = 4$, but does not hold for the graph $K_{3,4} - e$ where e is any edge dropped from $K_{3,4}$ for $r = 3$ and $s = 4$.

3. Does the inequality in Questions 1 and 2 hold for disconnected bipartite graphs? Why?

4. Establish the similar inequality for general graphs as in Problem 3 in Homework 3.

5. Reverse the roles of the r-tuple W and the vertex x in Question 1 so that now $x \in V_1$ and W is an r-tuple in V_2. Derive the new inequality as in Question 1 for the exclusion of a $K_{r,s}$ in $G_{m,n}$.

6. Try problems 5.2.23, 5.2.25 and 5.2.26 from the text by Douglas West.

7. Show that there is a bipartite subgraph $H(V_1 \cup V_2, E')$ of a graph $G(V, E)$ with at least half the edges of G, that is $|E'| \leq |E|$, where the two partites V_1 and V_2 differ by at most 1 in cardinality. Here $V_1 \cup V_2 = V$ but the subgraph is not an induced subgraph. (Lemma 9.6 and Exercise 9.16 in Pach and Agarwal.)

8. Show that the extention by arbitrary perfect graphs at each of the vertices of a given perfect graph G, again gives a perfect graph. (Lovasz 1972) (Exercise 43 of Chapter 5 in the text of Diestel.)

9. Exercise 41 from the text of Diestel.