Advanced graph theory: Tutorial 1: CS60047 Autumn August 12, 2022

1. Show that a k-regular bipartite graph has a perfect matching.

   [Solution: Corollary 5.4 in Bondy and Murty. Also Corollary 3.1.13 in West.]

2. Draw the 8-vertex 16-edge complement of the cube of 8 vertices of 12 edges.

3. Draw the Petersen graph P_{10} of $\binom{5}{2}$ vertices, where the vertices correspond to the 2-subsets of the first 5 natural numbers, and the edges correspond to disjoint pairs of such 2-subsets.

4. Find the size $\alpha(P_{10})$ of the largest independent set in the Petersen graph, and the smallest vertex cover of size $\beta(P_{10})$.

5. Show that a graph with girth 5 and minimum vertex degree $\delta \geq k$ has at least $k^2 + 1$ vertices.

   [If we have a girth 5 graph which is k-regular then taking any vertex v as the centre, we must have its k neighbours making up $k + 1$ vertices, where each of these k vertices neighbours $k - 1$ new neighbours, amounting to a total of at least $1+k+k(k-1) = 1+k+k^2-k = k^2+1$ distinct vertices. The equality holds for $k = 3$ in the Petersen graph.]

6. For $k = 2$ and $k = 3$ find examples of graphs of girth 5 and minimum vertex degree k with exactly $k^2 + 1$ vertices.

   [See Q 5 above for $k = 3$.]

7. Show that the Petersen graph is triangle-free.

8. Count the number of 5-cycles in the Petersen graph.

9. Show that all longest paths in a tree pass through a common vertex.

10. Show that this is not generally true for general graphs by giving an example.

11. Show that there are $n-k$ distinct paths of length k in a tree of diameter $2k - 3$.

    [See class slides.]