Advanced graph theory: Test 1: CS60047 Autumn 2022

5:05 pm to 6:15 pm; Time: 70 minutes; Maximum marks: 100

September 09, 2022

1. Given a connected simple undirected graph G that has k triangles in a linear sequence, interspersed and separated by $k-1$ cut edges, with a total of $3k$ vertices and $3k+(k-1)$ edges, find $\chi(G)$, $\alpha(G)$, $\beta'(G)$, and $\omega(G)$. [10 marks]

2. Consider a connected simple undirected graph $G(V, E)$ that has the following property. For every induced subgraph H of G, $|V(H)| \leq \alpha(H)\omega(H)$. Which other graphs derived from G will also satisfy the same property? Why? [10 marks]

   [Many graphs derived from G can satisfy the same property: the complement graph, the induced subgraphs of G and it complement, etc.]

3. Show that no 3-connected graph can have exactly seven vertices. [15 marks]

   [There is an error in the question, where instead of 7 vertices it should have been 7 edges. No 3-connected graph has exactly seven edges as $2e \geq n\kappa(G) \geq 3n$. This is because $2e \geq n\delta(G) \geq n\kappa(G) \geq 3n$. So, n must be at most $\frac{14}{3}$, i.e. at most 4. However, that can give only 6 edges.]

4. Show that in a triangle-free graph G, the number of edges is at most $\alpha(G)\beta(G)$. [10 marks]

   [Each vertex in the minimum vertex cover can connect with vertices that form an independent set if there are no triangles.]

5. For a connected graph show that $2\alpha'(G) \geq \beta(G) \geq \alpha'(G)$. Characterize a family of graphs with $n > 3$ vertices where $\beta(G)$ approaches $2\alpha(G)$ as n grows. [8+7 marks]

[Each maximum matching edge needs to be covered. Other edges can be covered by the other vertex of the matching edge. Note that K_n has a matching of size $\frac{n}{2}$ and a vertex cover of size $n-1$.]

6. Show that in an undirected connected graph G, $\alpha(G) \geq \frac{|V|}{\Delta(G)+1}$ where $\Delta(G)$ is the maximum vertex degree. [15 marks]

   [Since each of the $\alpha(G)$ vertices of the maximum independent set would have at most $\Delta(G)$ neighbours, with edges connecting to these neighbours landing only in the minimum vertex cover, we have $|V| - \alpha(G) \leq \alpha(G)\Delta(G)$, whence $\alpha(G) \geq \frac{|V|}{\Delta(G)+1}$.]

7. Show that a tree can have at most one perfect matching. [10 marks]

   [A graph with two distinct perfect matchings would have cycles in the symmetric difference of the two matchings.]

8. Suppose a tree T has a perfect matching. Then show that $o(T \setminus \{v\}) = 1$ for every vertex v of T. [15 marks]

   [If there were no odd components then the total number of vertices is odd. We know that $o(T \setminus \{v\}) \leq 1$. Why?]