Advanced graph theory: Mid-Semester Examination : CS60047 Autumn 2022

9:00 am to 11:00 am: Marks: 100
Do the first question and any six of the remaining questions.
You are free to either use the hints or solve the problems independently.

September 27, 2022

1. Show that in a graph G that has no isolated vertices and has maximum vertex degree $\Delta(G)$, $\beta(G) \geq \frac{|V(G)|}{\Delta(G)+1}$, where $V(G)$ is the vertex set of G. (10 marks)

2. Given a tree T show that its complement $\bar{T}$ is a perfect graph. (15 marks) [Hint: This follows from the perfect graph theorem. However, you must establish this by using strong induction, Konig-Egervary theorem and Gallai's theorem.]

3. Assume that two chordal graphs can be combined and united by pasting them along their common complete subgraph to result in another chordal graph. Also, complete subgraphs are chordal. Assume further that any chordal graph can be thus constructed by combining two smaller chordal graphs by pasting them along their common complete graph, starting with complete subgraphs. Based on these assumptions, whereby any chordal graph can be constructed as mentioned, show that chordal graphs are perfect. (15 marks)

   [Hints: Since any chordal graph G can be constructed by pasting some two chordal graphs G_1 and G_2 along their common complete subgraph S, and that by the induction hypothesis, we may assume that G_1 and G_2 are perfect, all we need to do is show that G is perfect by proving that for every induced subgraph H of G, $\chi(H) \leq \omega(H)$. Note that induced subgraphs of G may overlap with G_1 and G_2, whereas all induced subgraphs of each of G_1 or G_2 are perfect because G_1 and G_2 are perfect by the induction hypothesis.]

4. Suppose a simple graph has a quadrilateral then show that it has at least $\frac{n}{4}(1 + \sqrt{(4n-3)})$ edges. (15 marks)

   [Hints: For any two vertices x and y, we cannot have two common neighbours if we wish to exclude C_4 or a quadrilateral. So, an $\binom{n}{2}$ upper bound is immediate on the number of such pairs that have a common neighbour. A vertex z having two neighbours x and y can have more such pairs of neighbours if its degree $d(z)$ is more than two. You can use Jensen's inequality suitably.]

5. Show that a 3-regular graph with at most two cut edges has a perfect matching. (15 marks)

   [Hint: For the sake of contradiction assume that there is no perfect matching and S is the bad set. Find whether the parities of the set S and $o(G-S)$ are same or different and write the violated Tutte's condition.

Now plug in the requirements that there are at most two cut edges to derive a contradiction.]

6. We say that a graph is *randomly traceable* if a spanning path always results upon starting at any vertex of G and then successively proceeding to any vertex not yet chosen until no new vertices are available. Show that a graph G with $2n \geq 4$ vertices exists, that is randomly traceable, but which is neither a C_{2n} nor a K_{2n}.

   We say that a graph is *arbitrarily traversable* from a vertex v_0, if starting a traversal at v_0, we traverse any incident edge, and on arriving at a vertex u, we depart from u by traversing any incident edge not yet used, and continue until no new edges remain. Show that if a graph G is arbitrarily traversable from a vertex v_0 then v_0 has maximum degree. (8+7 marks)

7. We wish to show that intersections of subtrees of a tree obey the Helly property, whereby the set $\mathcal{S} = \{T | T \in \mathcal{T}\}$ of pairwise intersecting subtrees of a tree $\mathcal{T}$ also has a non-empty intersection $\cap_{T \in \mathcal{S}} T$. Complete the following argument for establishing this Helly property for intersecting subtrees of a tree.
   (15 marks)

   [Hints: Suppose we use induction on the number k of subtrees of an n-vertex tree $\mathcal{T}$. Conisder k subtrees $T_1, T_2, ..., T_k$ of $\mathcal{T}$ which intersect pairwise. For the sake of contradiction we assume that they do not have a common intersection. However, by the induction hypothesis $T_1, T_2, ..., T_{k-1}$ intersect in say a subtree T_0. As T_k misses T_0 let us find a connecting path P from T_0 to T_k with a vertex $x \in P \cap T_k$ and a vertex y adjacent to x on P closer to T_0. Now $\mathcal{T} - xy$ has connected components where the edge xy separates T_0 from T_k.]

8. Show that for any graph G, $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraphs of G. Also, show that $\chi(G) \leq n - \alpha(G) + 1$, where n is the number of vertices of G. (8+7 marks)

   [Hint: Let $\chi(G) = c \geq 2$. If H is any smallest induced subgraph such that $\chi(H) = c$, then show that for all induced subgraphs H' of H, we have $\max \delta(H') \geq \delta(H) \geq c - 1$.]

9. Use Hall's theorem to show that $\beta(G) \leq \alpha'(G)$ in a bipartite graph G. (15 marks) [Show that for any minimum cardinality vertex cover in G, we can demonstrate a matching of the same size.]

10. Show that a graph G is perfect if and only if it has the property that every induced subgraph H contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$. (15 marks) [Hint: Use induction.]

11. A graph G is such that in every induced subgraph H each maximal independent set of H meets every maximal complete subgraph of H. Show that such a graph G is perfect. (15 marks) [Hint: Use induction.]