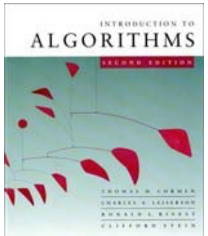
CS60020: Foundations of Algorithm Design and Machine Learning

Sourangshu Bhattacharya

MINIMUM SPANNING TREES

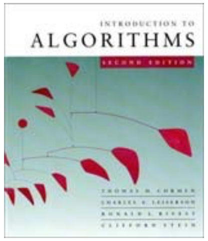


Minimum spanning trees

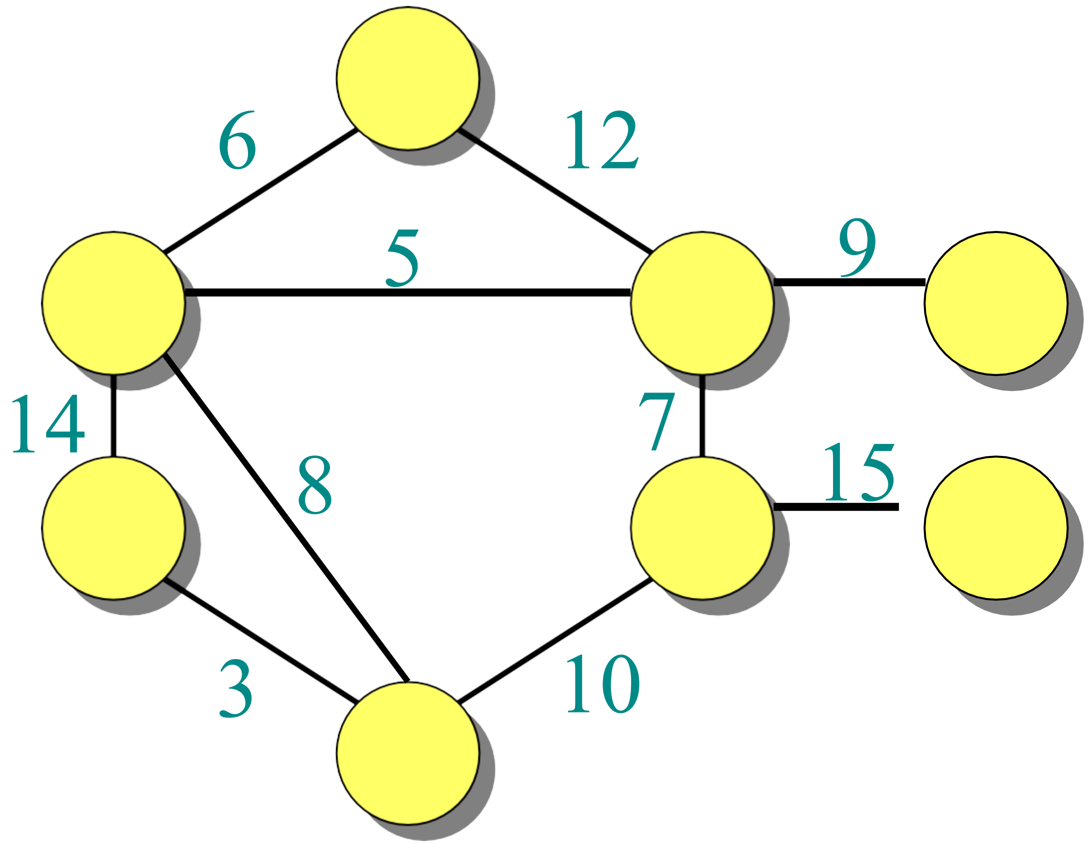
Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

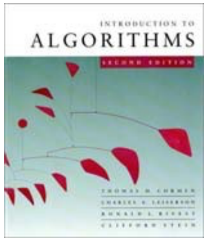
Output: A ***spanning tree*** T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u, v) .$$



Example of MST





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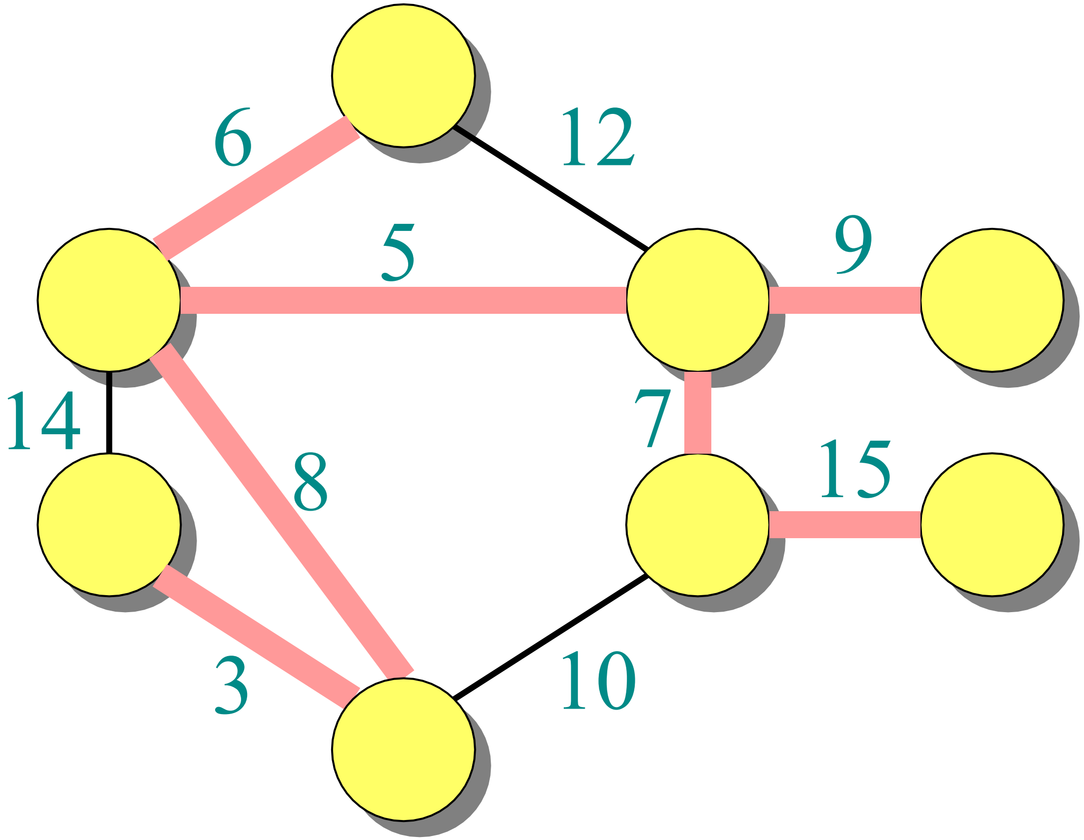
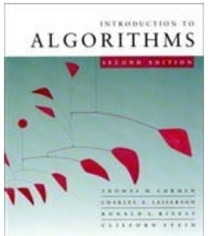
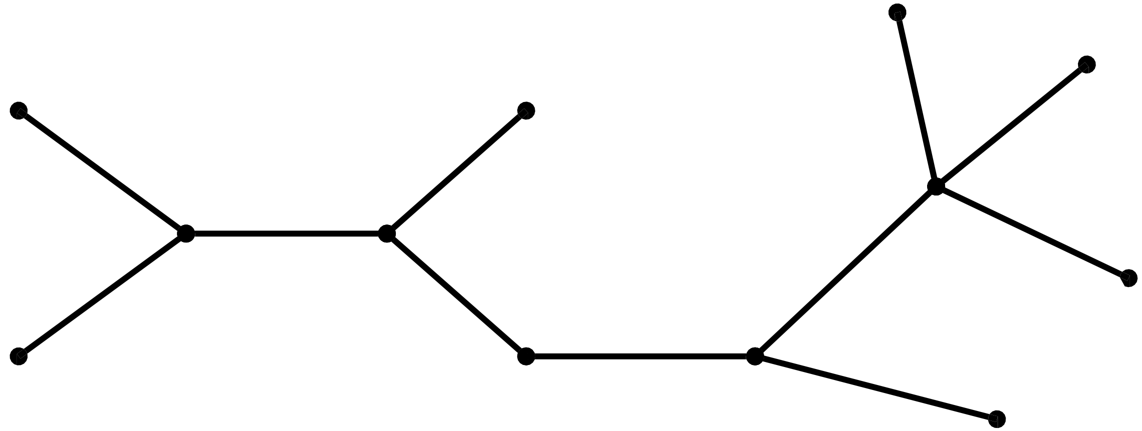


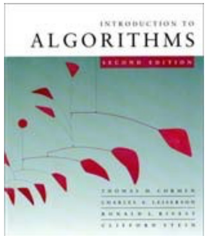


Optimal substructure

MST *T*:

(Other edges of *G* are not shown.)





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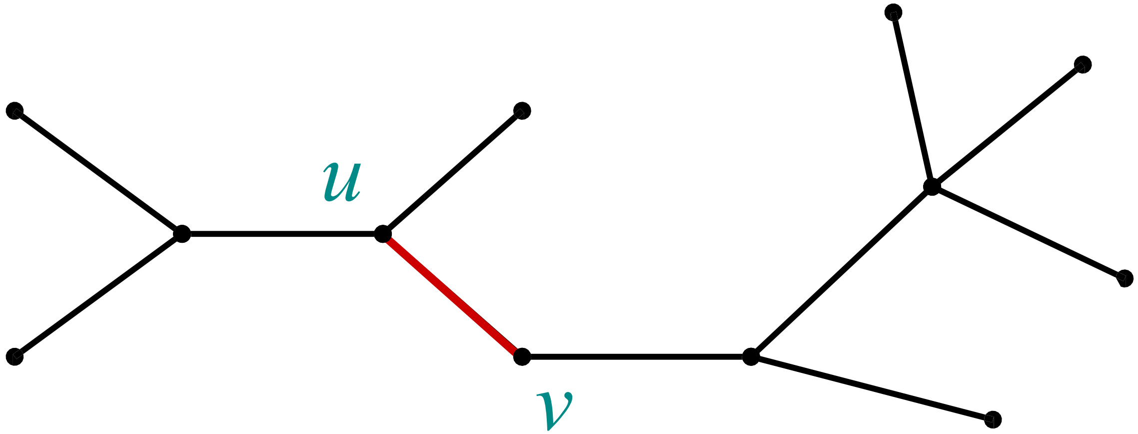
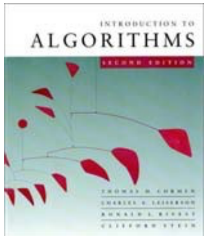


Remove any edge $(u, v) \in T$.



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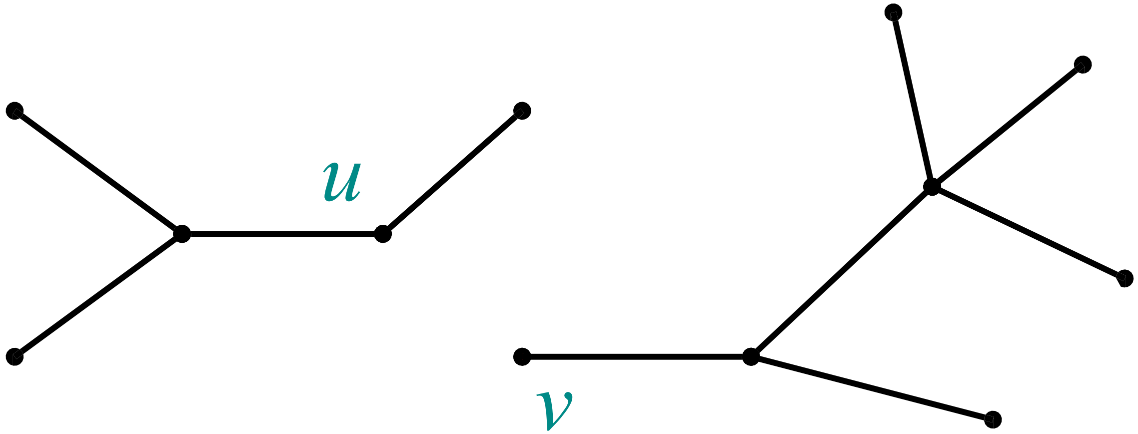

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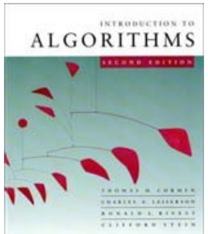

Optimal substructure

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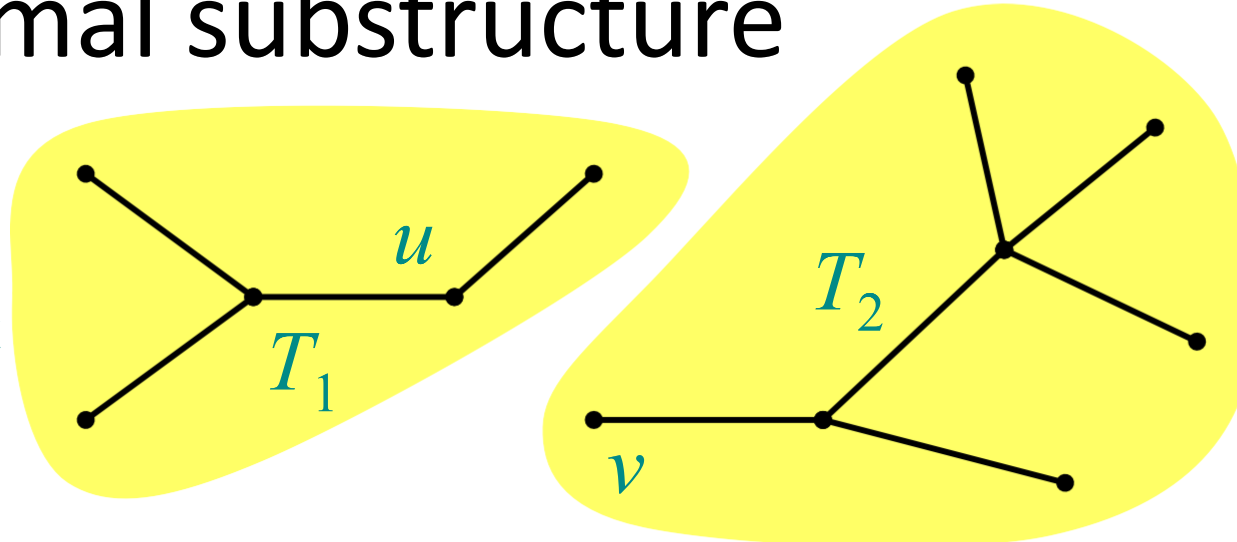
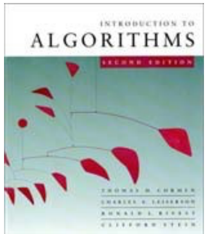


Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.



Optimal substructure

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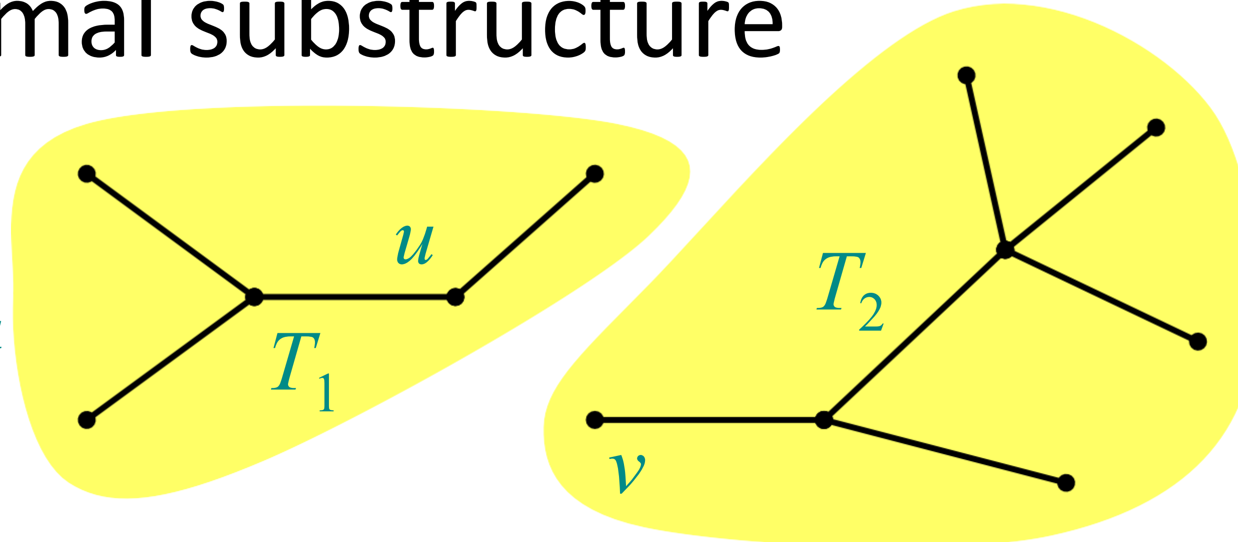
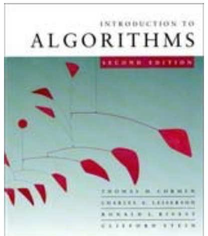


Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.

Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G ***induced*** by the vertices of T_1:

$$V_1 = \text{vertices of } T_1,$$
$$E_1 = \{ (x, y) \in E : x, y \in V_1 \}.$$

Similarly for T_2.

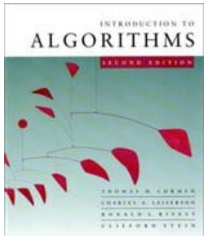


Proof of optimal substructure

Proof. Cut and paste:

$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G. □



Hallmark for "greedy" algorithms

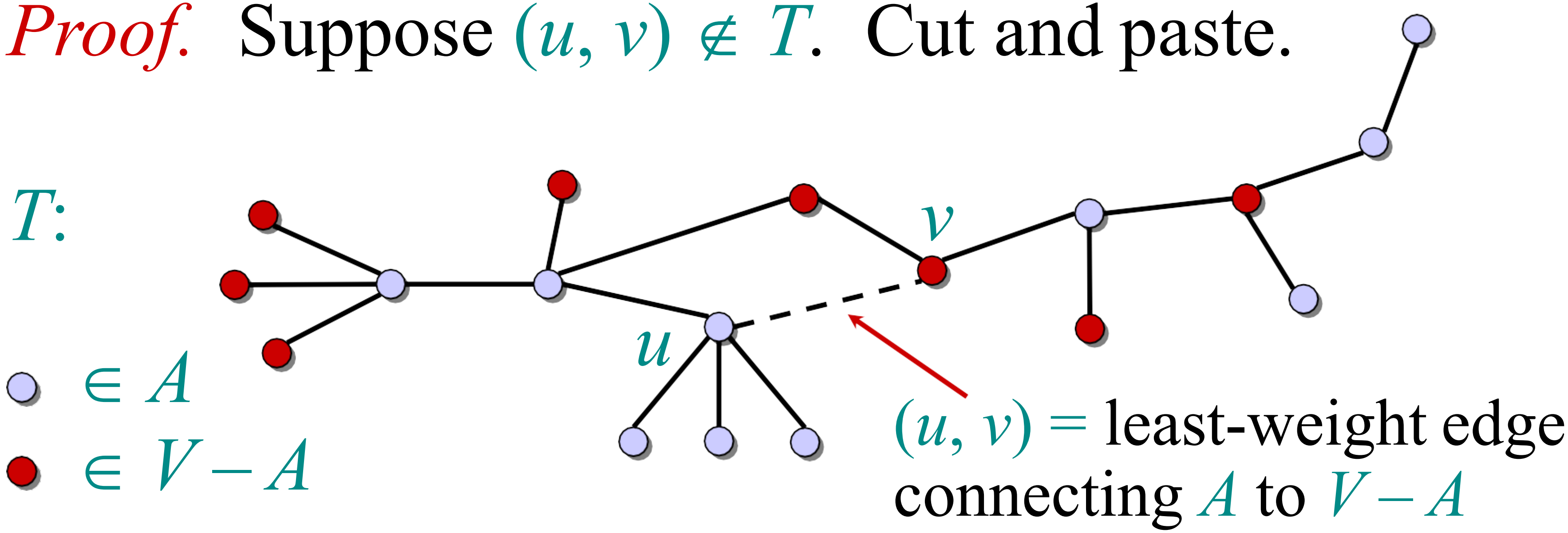
Greedy-choice property

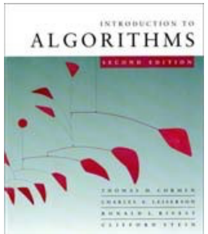
A locally optimal choice is globally optimal.

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.





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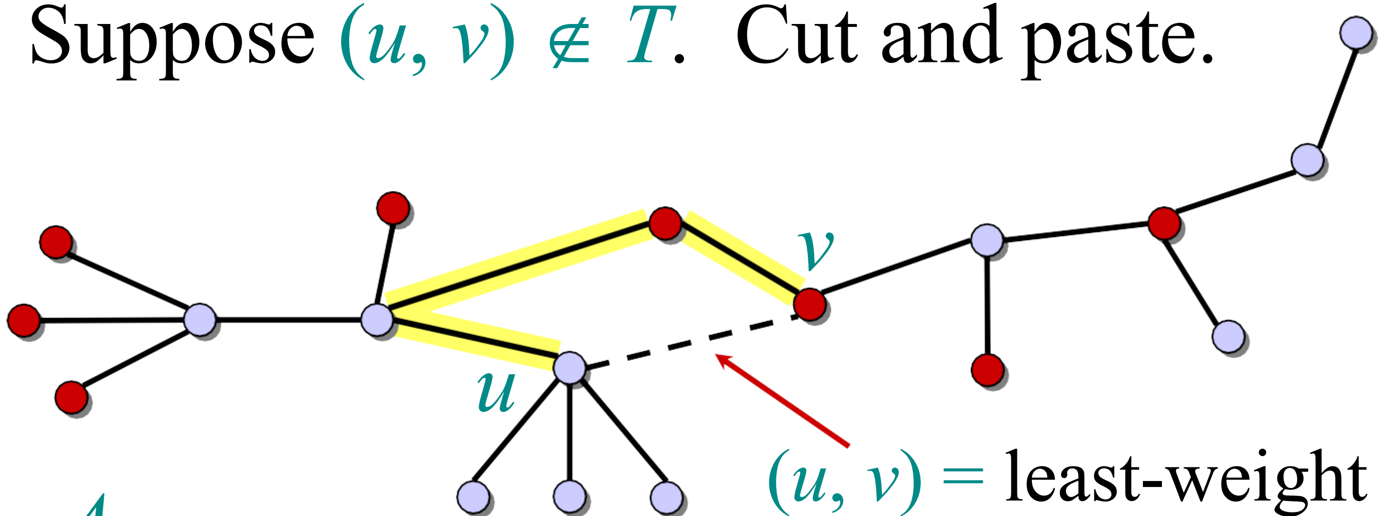
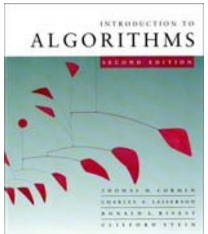


○ $\in A$

● $\in V - A$

(u, v) = least-weight edge connecting A to $V - A$

Consider the unique simple path from u to v in T.



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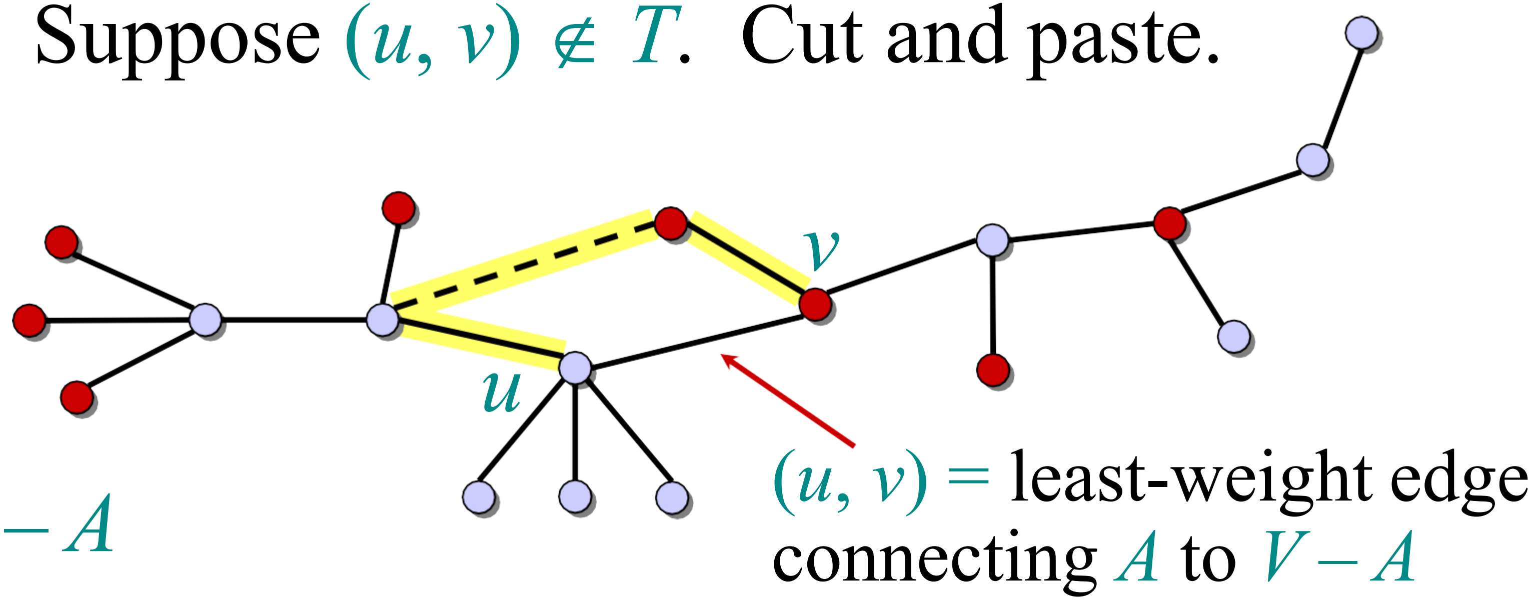
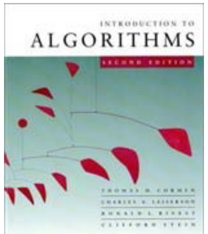


● $\in A$

● $\in V - A$

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.



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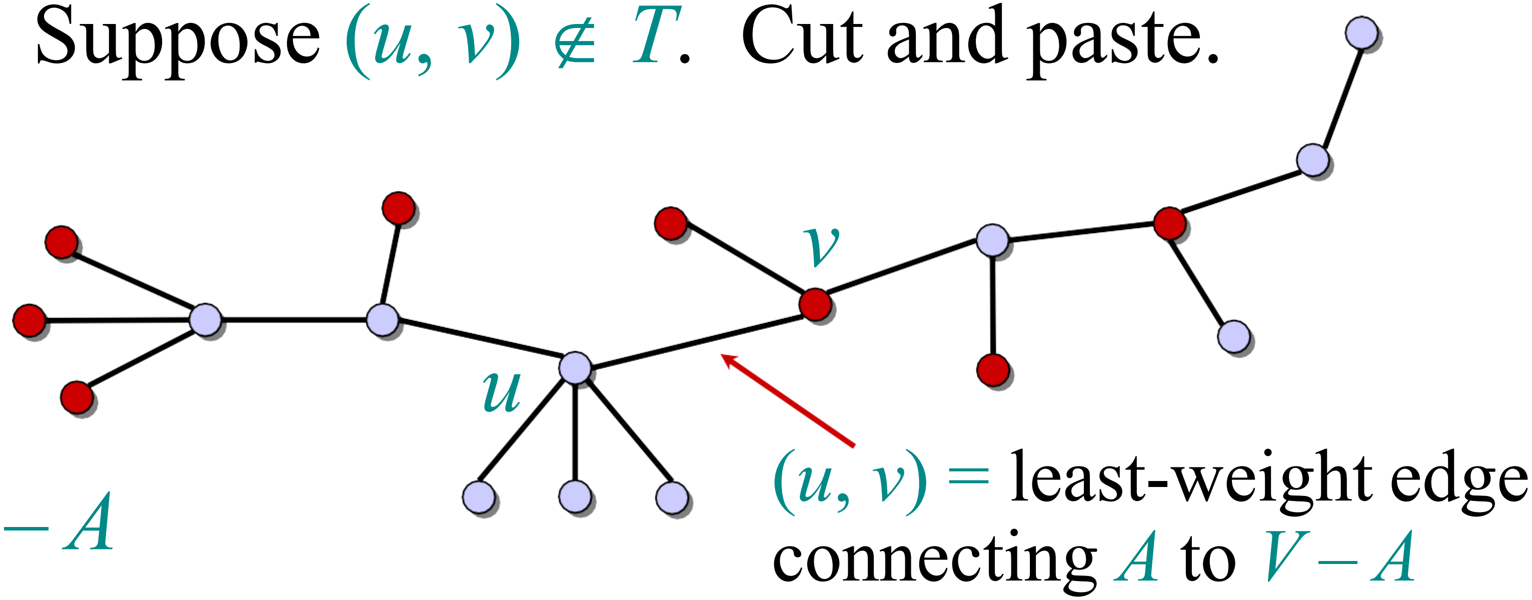


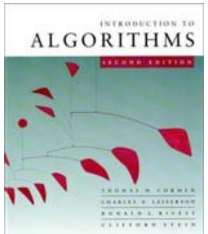
● $\in A$

● $\in V - A$

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

A lighter-weight spanning tree than T results. □

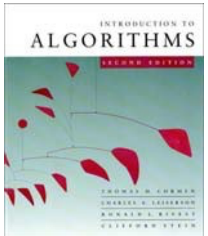


Prim's algorithm

IDEA: Maintain $V - A$ as a priority queue Q. Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A.

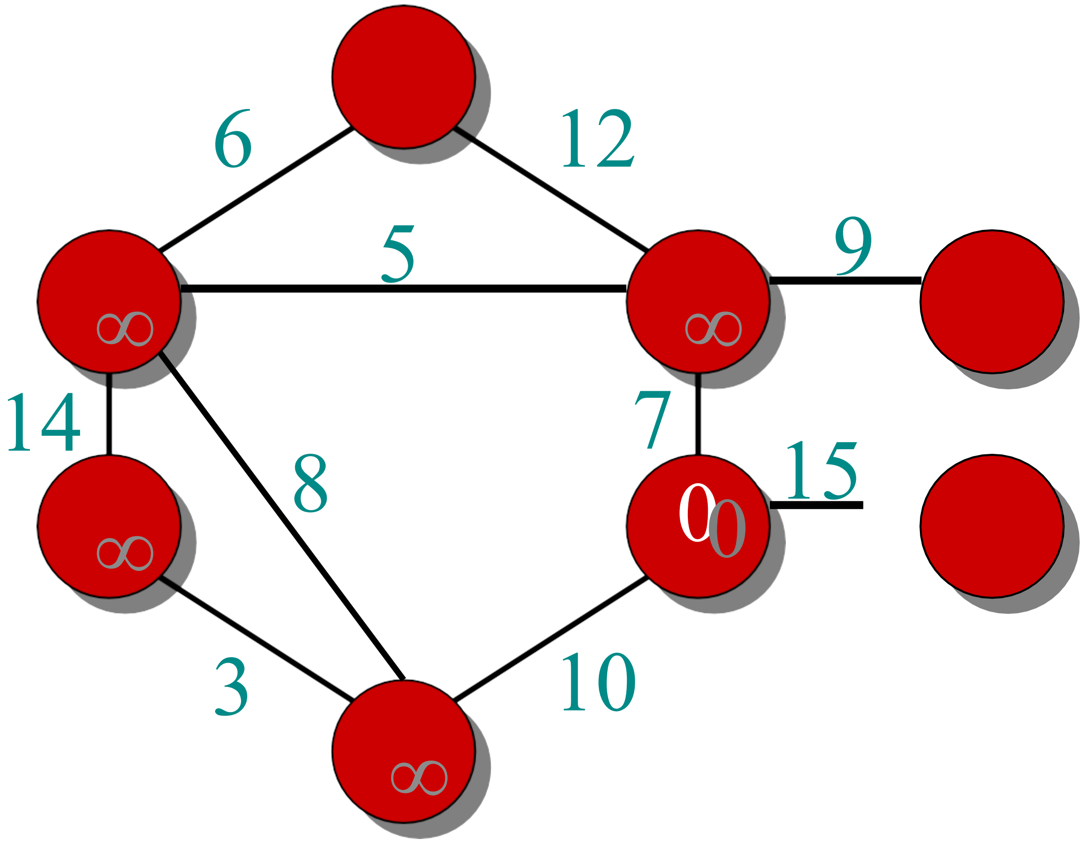
$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        for each $v \in Adj[u]$
            do if $v \in Q$ and $w(u, v) < key[v]$
                then $key[v] \leftarrow w(u, v)$ ⊳ DECREASE-KEY
                    $\pi[v] \leftarrow u$

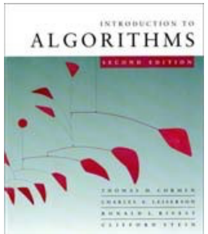
At the end, $\{(v, \pi[v])\}$ forms the MST.



Example of Prim's algorithm

- $\in A$
- $\in V - A$





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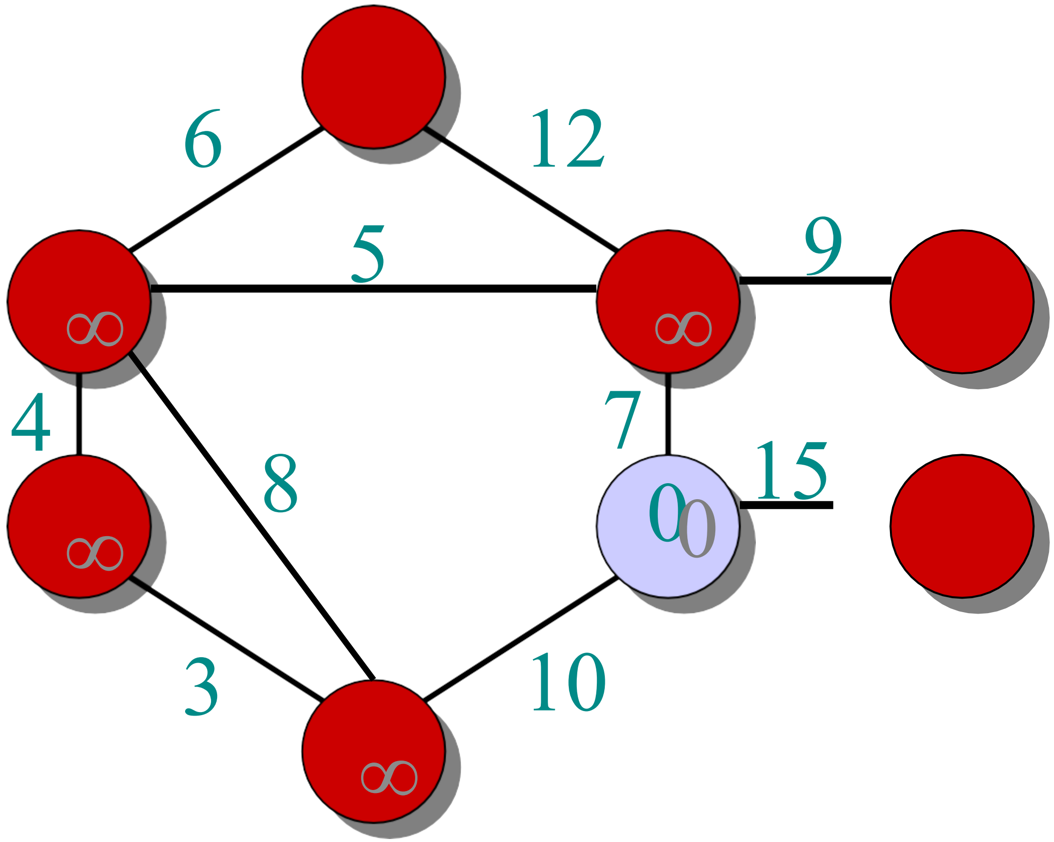

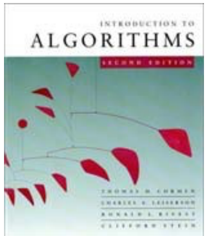

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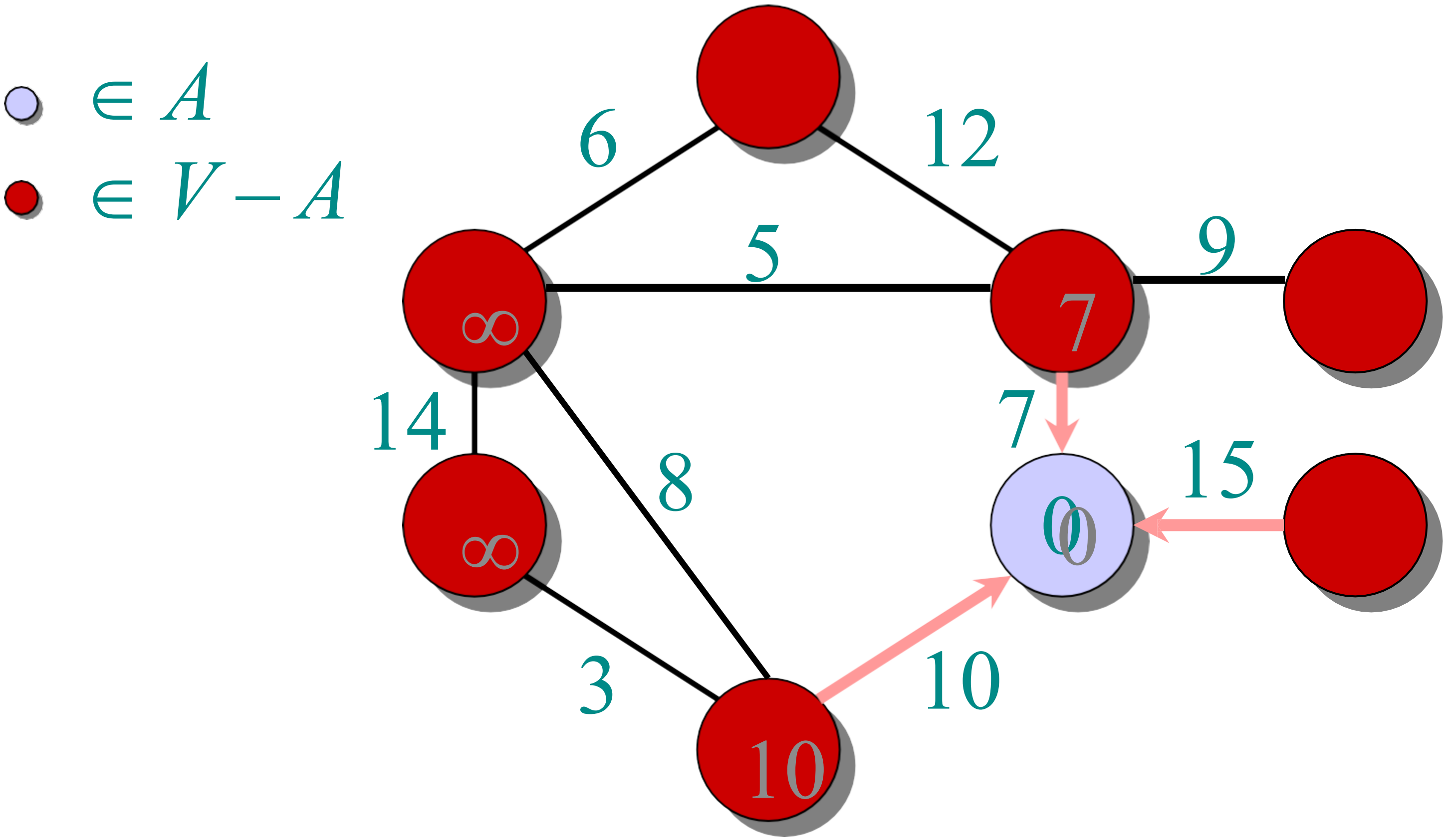

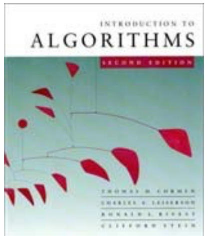

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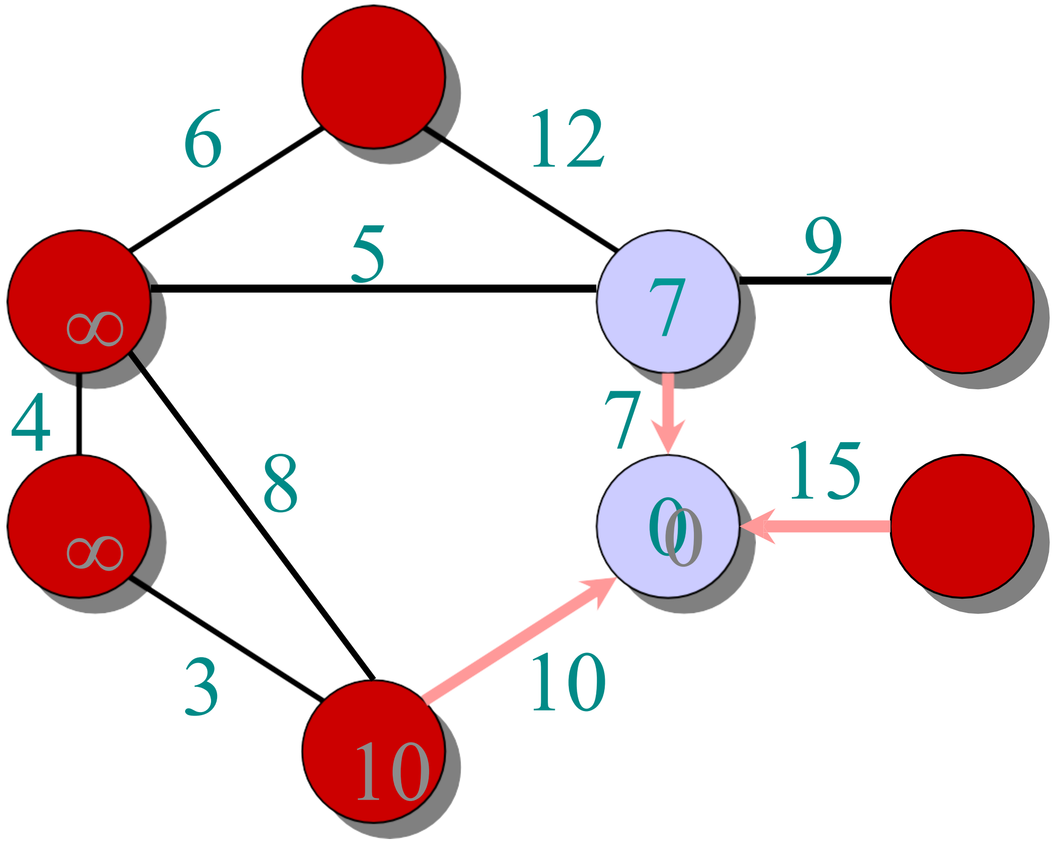

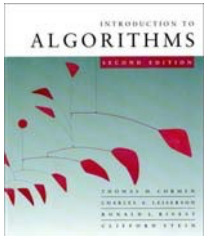

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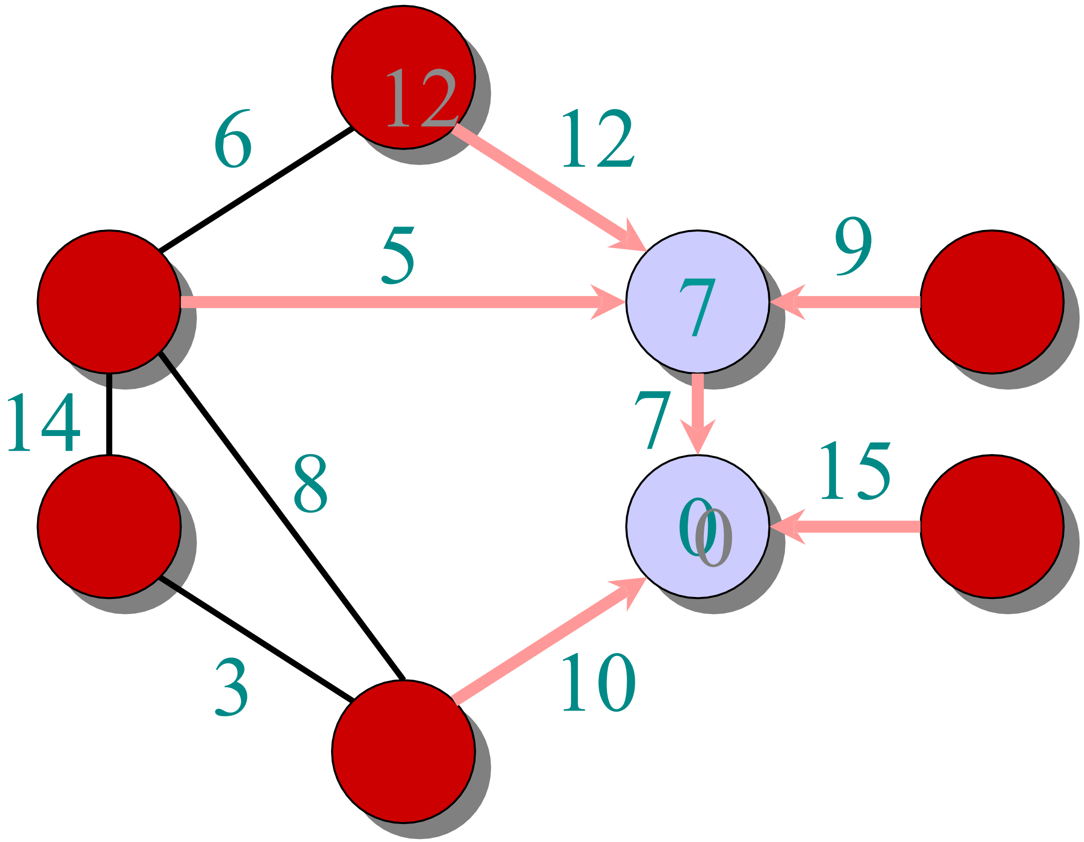

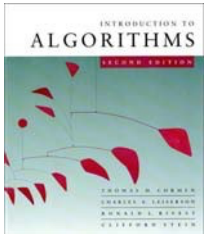

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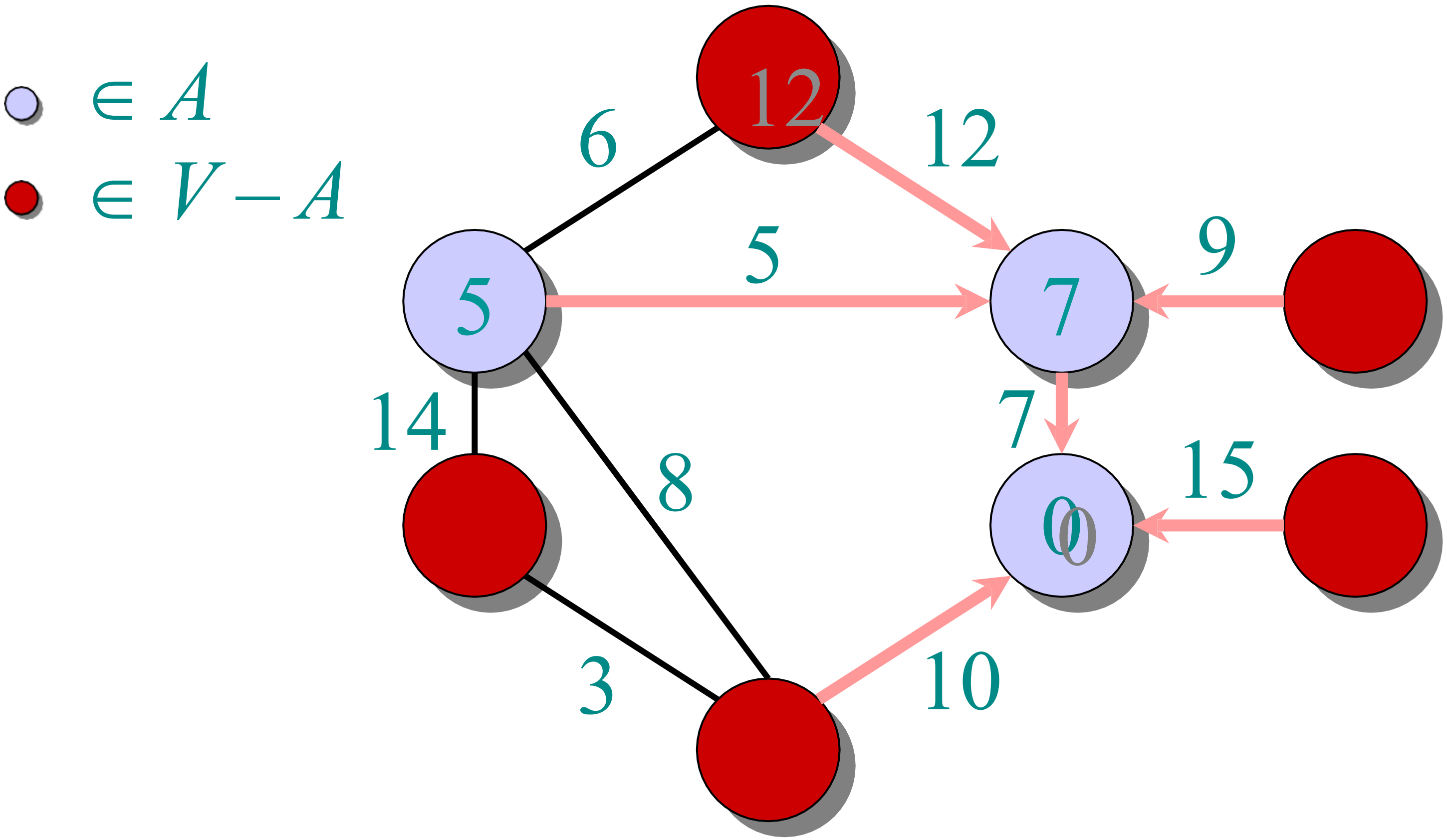

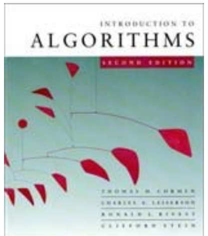

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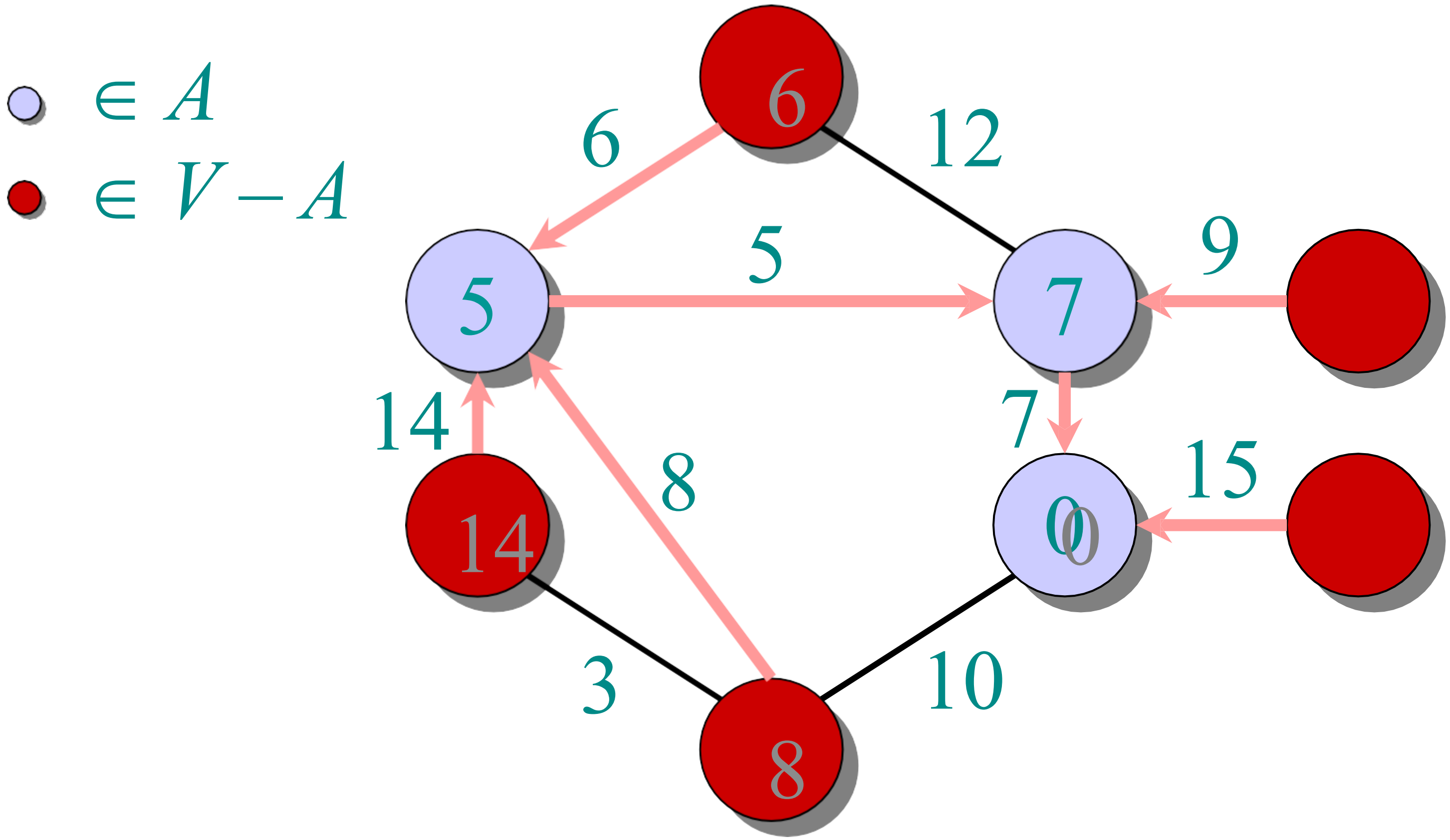

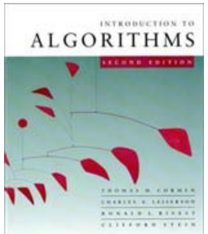

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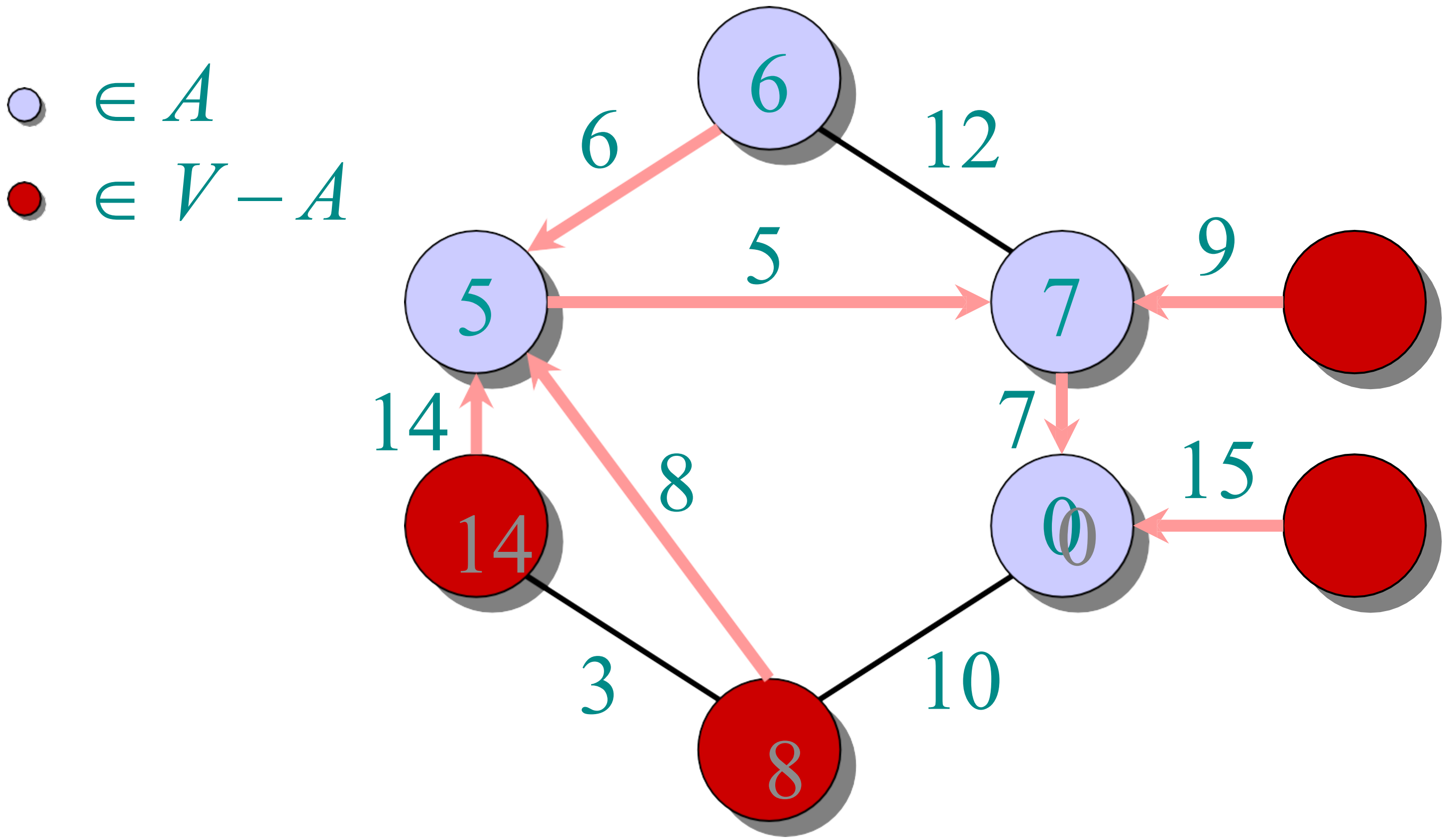

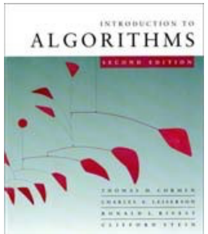

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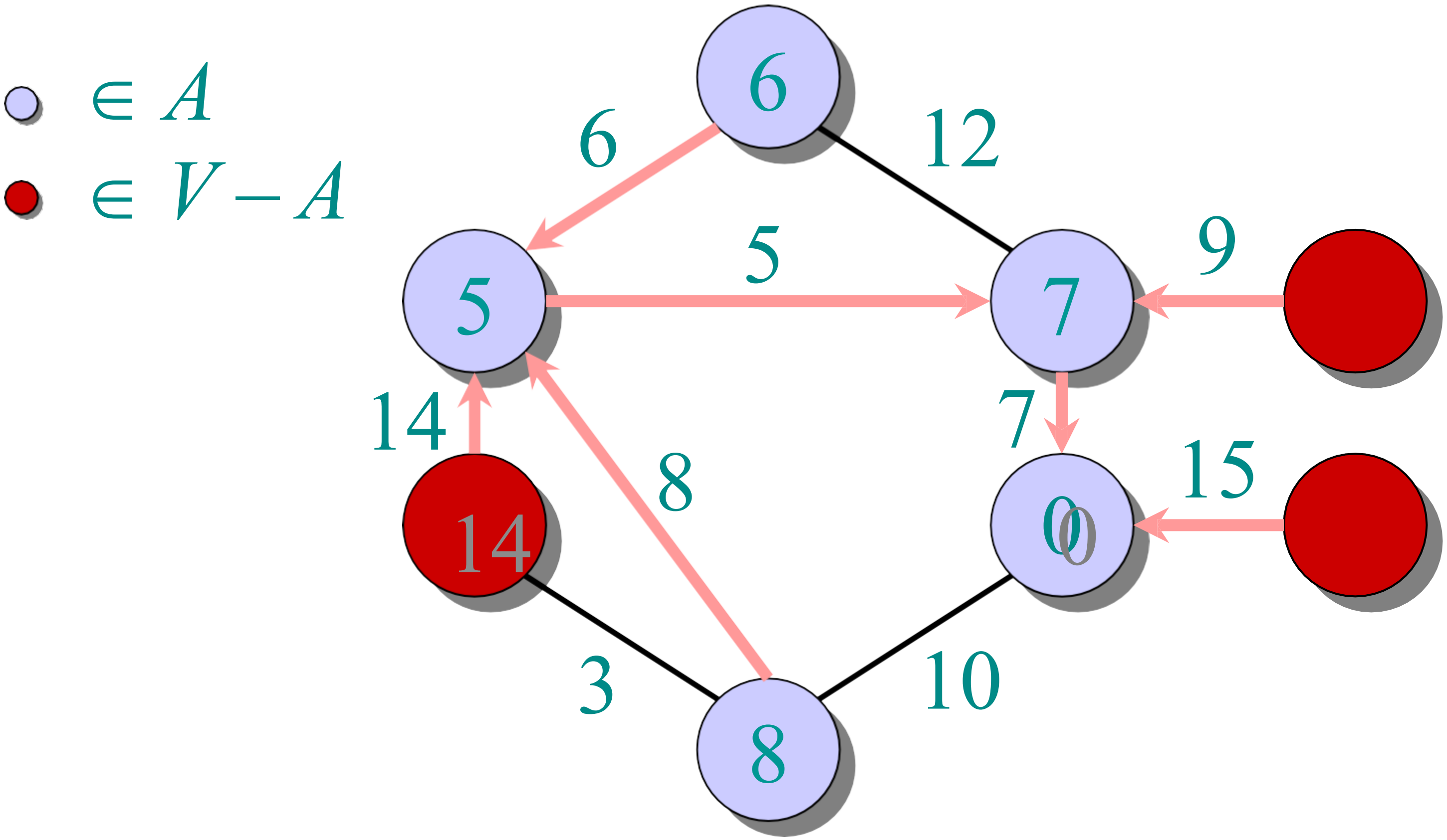

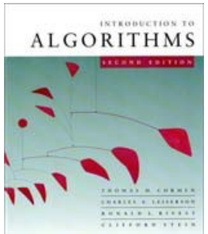

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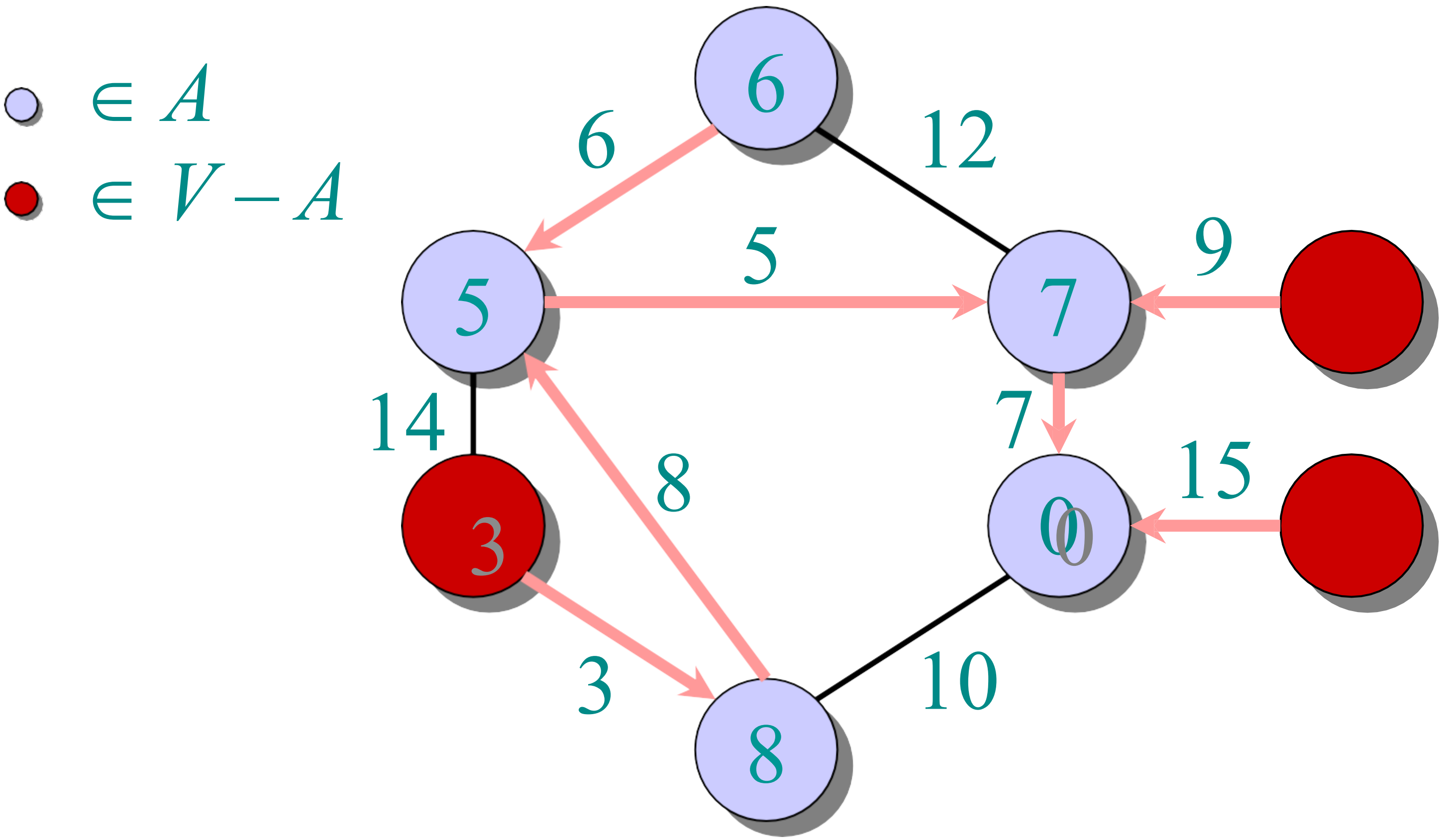

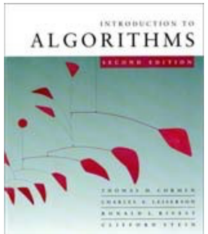

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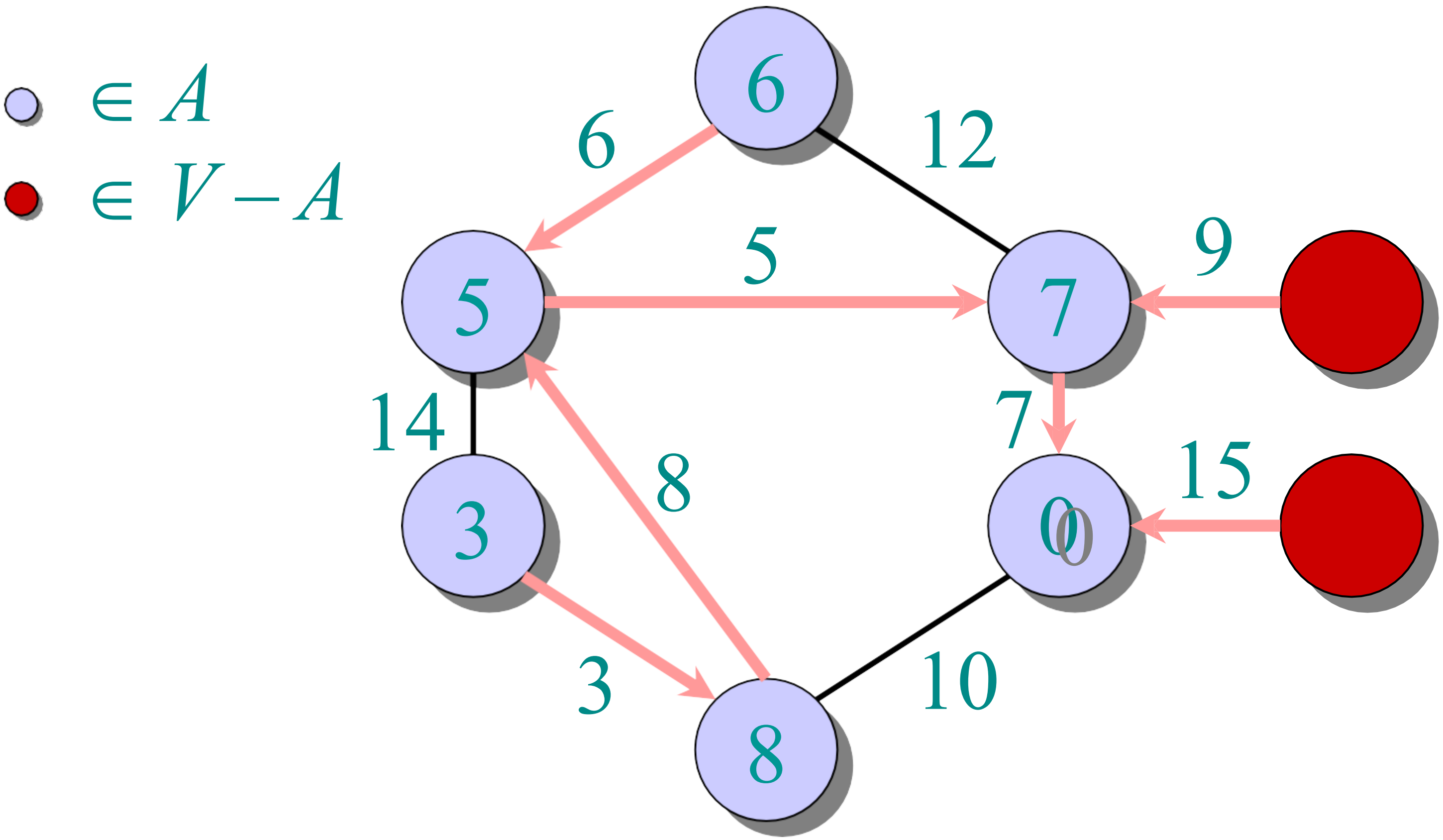

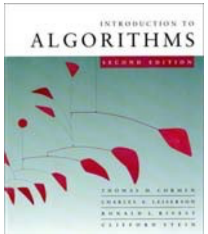

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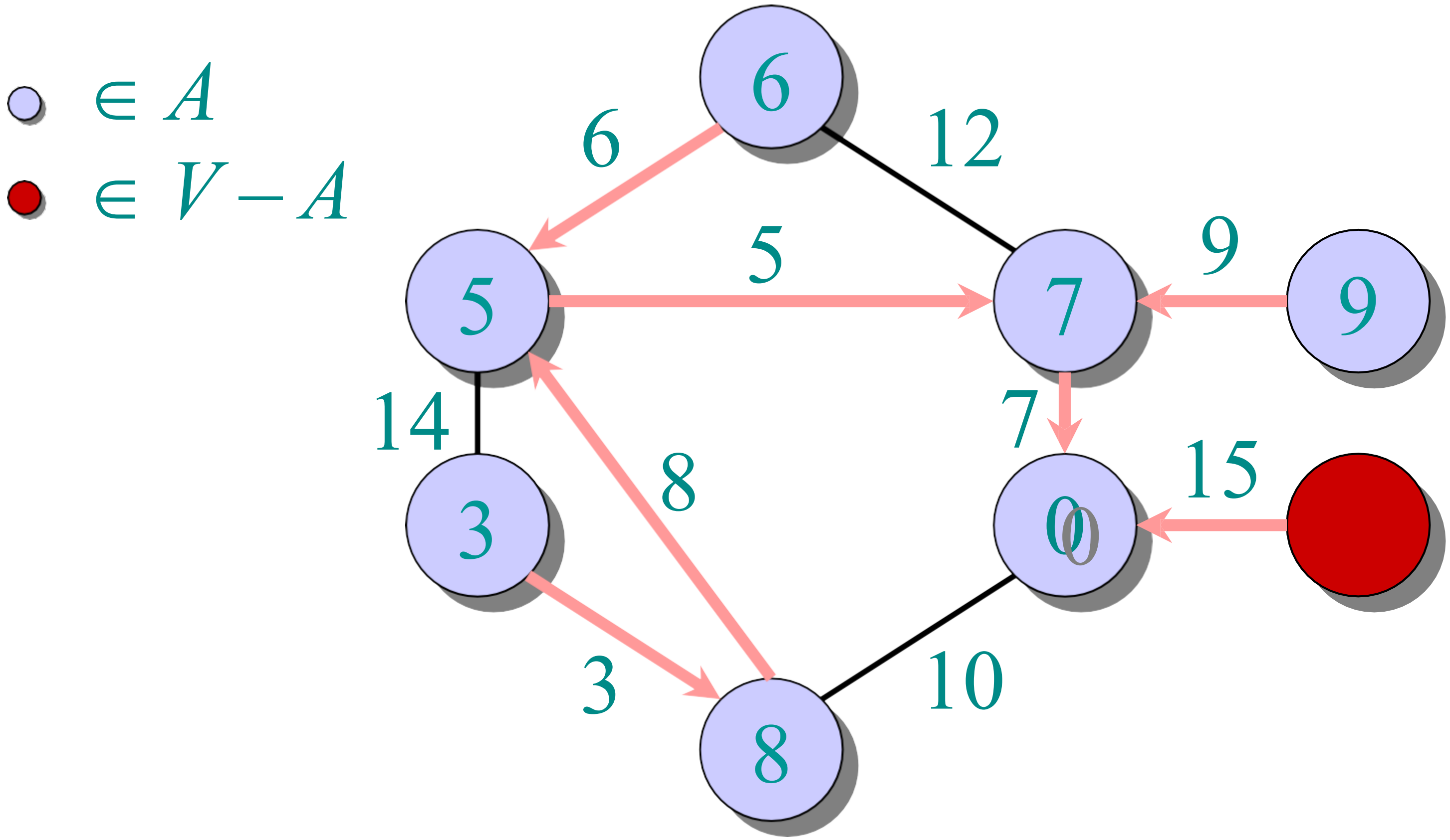

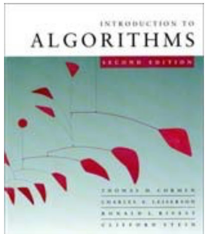

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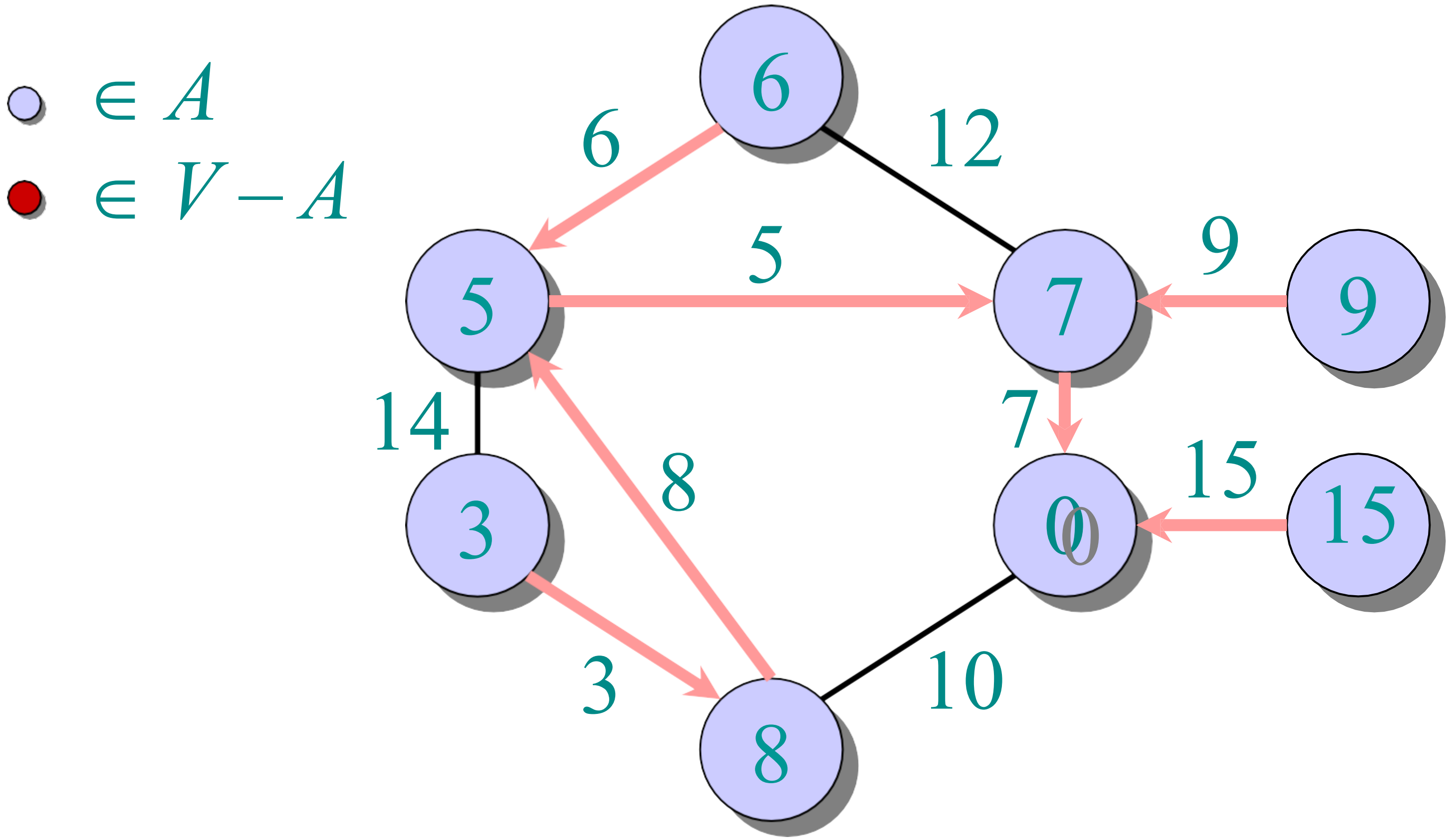

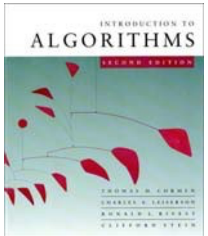

Analysis of Prim

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    for each $v \in Adj[u]$

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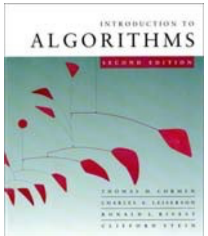

Analysis of Prim

$\Theta(V)$ total:

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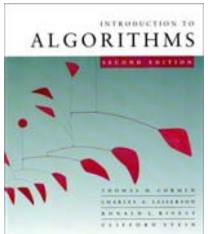

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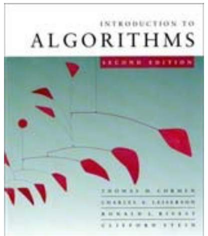

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$|V|$ times:

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$degree(u)$ times:

- **for each** $v \in Adj[u]$
- **do if** $v \in Q$ **and** $w(u, v) < key[v]$
- **then** $key[v] \leftarrow w(u, v)$
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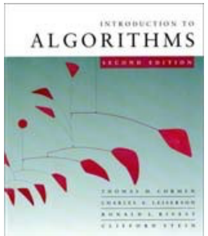

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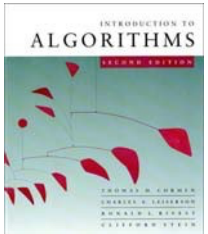
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.



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- $key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times:

while $Q \neq \emptyset$

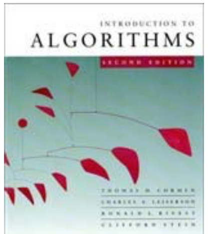
  do $u \leftarrow$ EXTRACT-MIN(Q)

  $degree(u)$ times:

  - **for each** $v \in Adj[u]$
  - **do if** $v \in Q$ and $w(u, v) < key[v]$
    - **then** $key[v] \leftarrow w(u, v)$
    - $\pi[v] \leftarrow u$

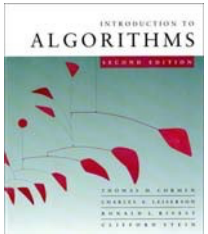
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$



Analysis of Prim (continued)

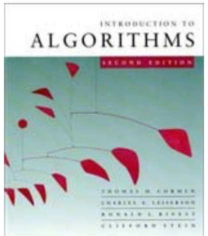
Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$



Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\textsc{Extract-Min}} + \Theta(E)\cdot T_{\textsc{Decrease-Key}}$$

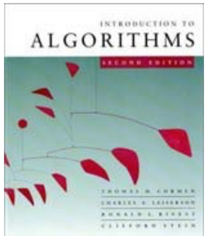
Q	$T_{\textsc{Extract-Min}}$	$T_{\textsc{Decrease-Key}}$	Total



Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

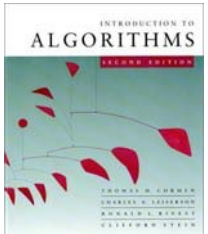
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$



Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

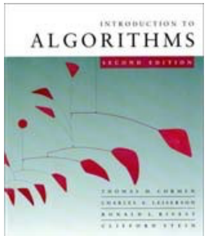
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$



Analysis of Prim (continued)

Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case



MST algorithms

Kruskal's algorithm (see CLRS):

- Uses the ***disjoint-set data structure*** (Lecture 10).
- Running time $= O(E \lg V)$.

Best to date:

- Karger, Klein, and Tarjan [1993].
- Randomized algorithm.
- $O(V + E)$ expected time.