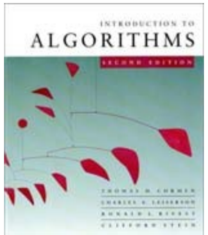
CS60020: Foundations of Algorithm Design and Machine Learning

Sourangshu Bhattacharya

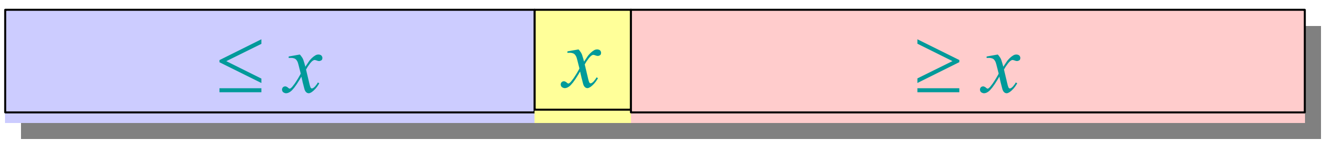
QUICKSORT



Divide and conquer

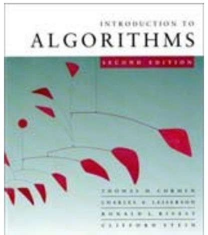
Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*



Partitioning subroutine

PARTITION(A, p, q) ⊲ $A[p\,.\,.\,q]$

$x \leftarrow A[p]$ ⊲ pivot $= A[p]$

$i \leftarrow p$

for $j \leftarrow p + 1$ **to** q

do if $A[j] \leq x$

then $i \leftarrow i + 1$

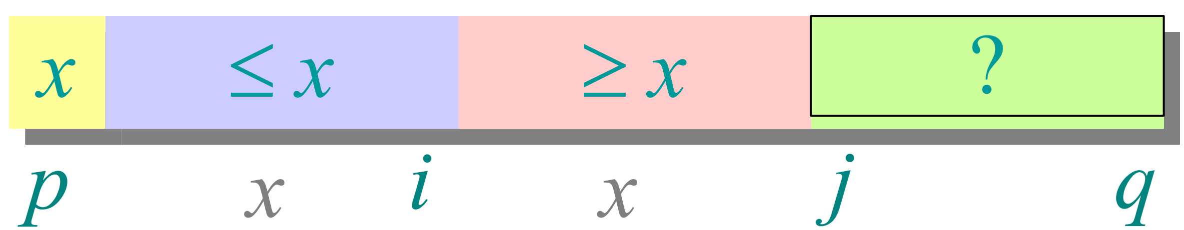
exchange $A[i] \leftrightarrow A[j]$

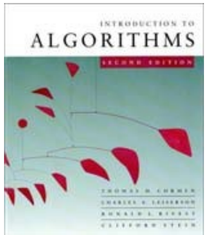
exchange $A[p] \leftrightarrow A[i]$

return i

Running Time $= O(n)$ for n elements

Invariant:





Pseudocode for quicksort

QUICKSORT(A, p, r)

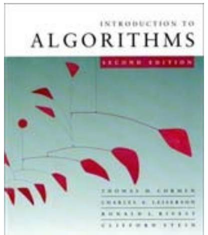
    if $p < r$

        then $q \leftarrow$ PARTITION(A, p, r)

            QUICKSORT$(A, p, q-1)$

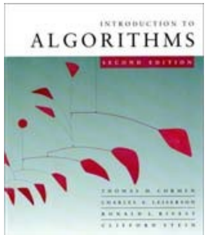
            QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$



Analysis of quicksort

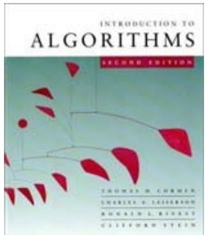
- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.



Worst-case of quicksort

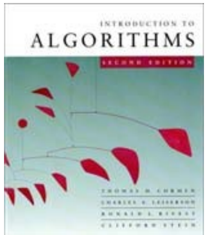
- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{\textbf{(arithmetic series)}}
\end{aligned}
$$



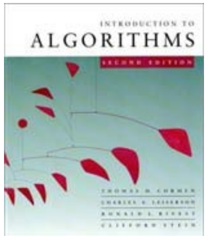
Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn \quad T(n)$$



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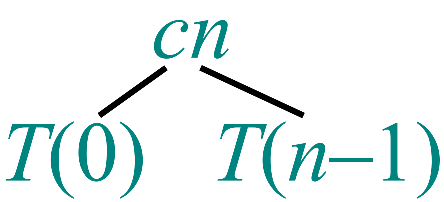

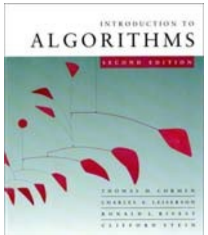

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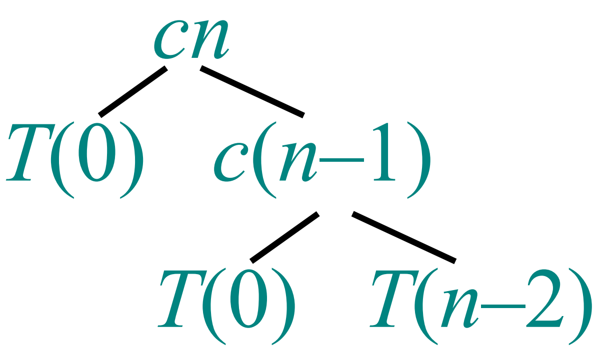

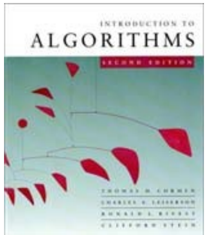

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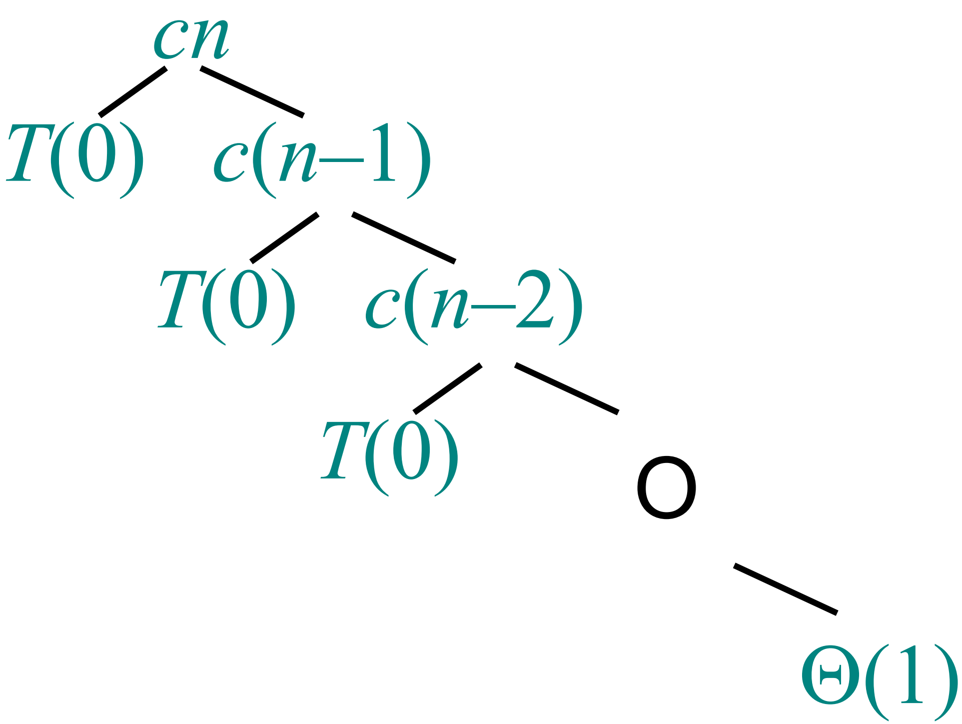

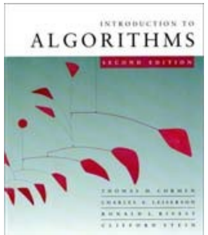

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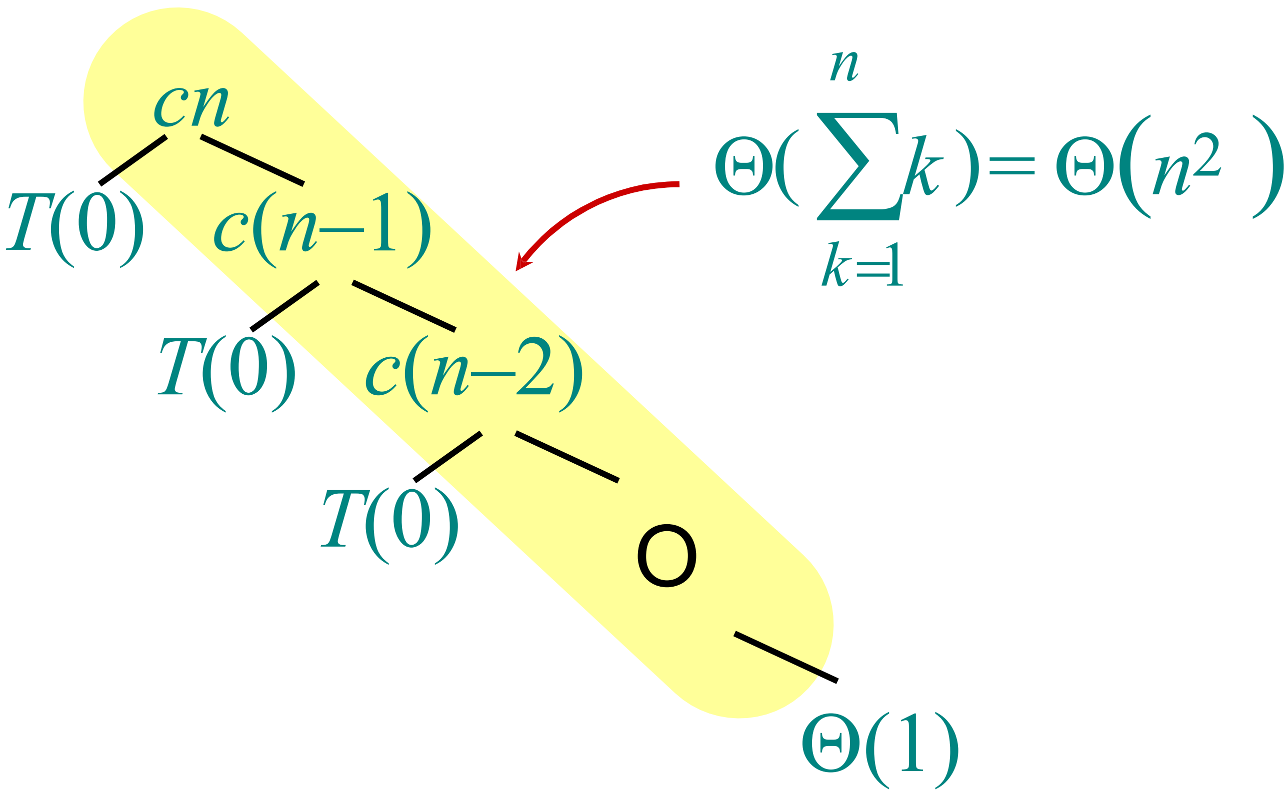

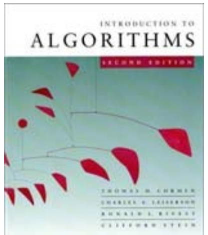

Worst-case recursion tree

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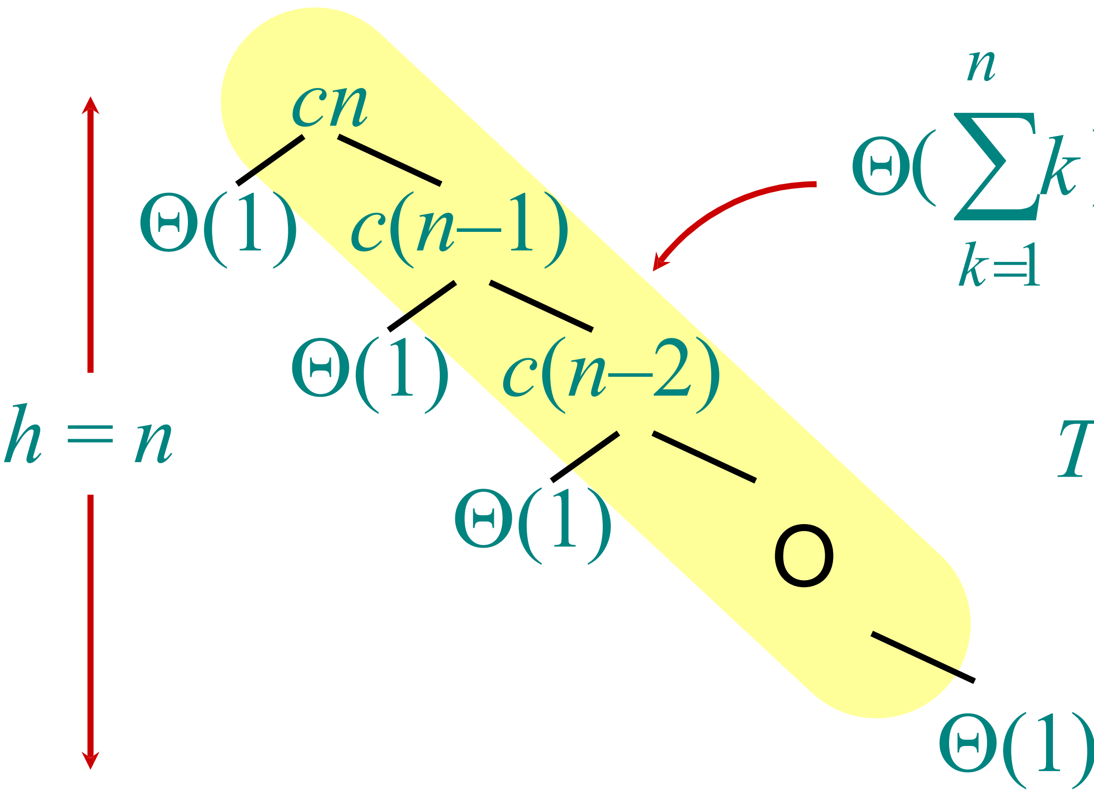
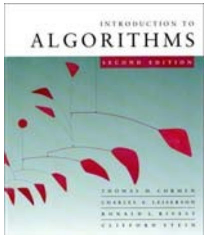


$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$



Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

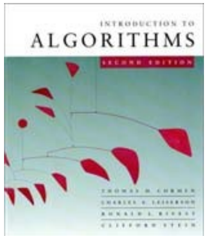
$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

(same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10} n\right) + T\left(\tfrac{9}{10} n\right) + \Theta(n)$$

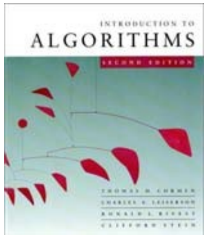
What is the solution to this recurrence?



Analysis of "almost-best" case

$$cn$$

$$T\left(\frac{1}{10}n\right) \qquad T\left(\frac{9}{10}n\right)$$



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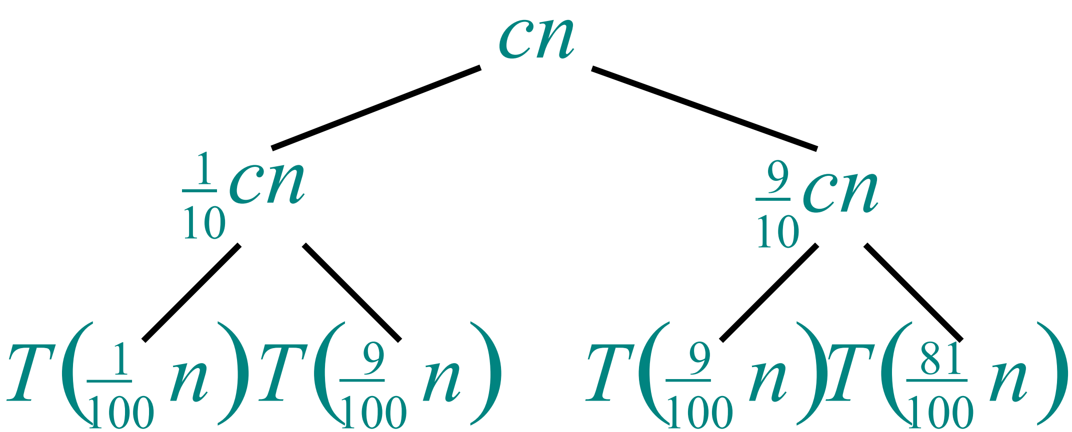

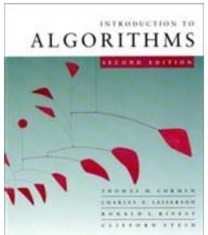

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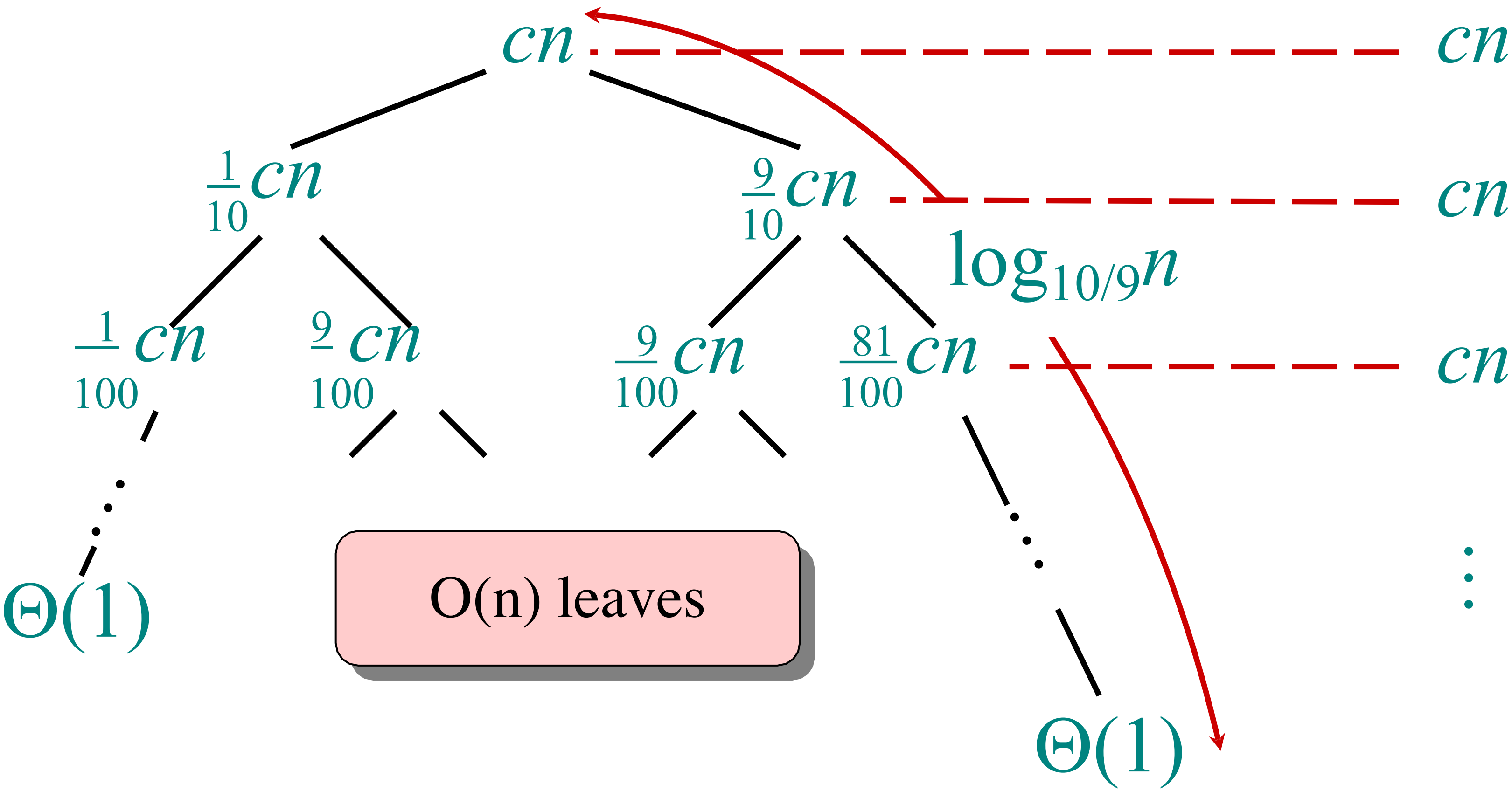

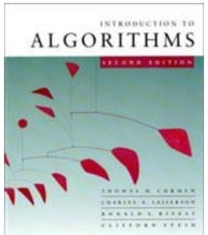

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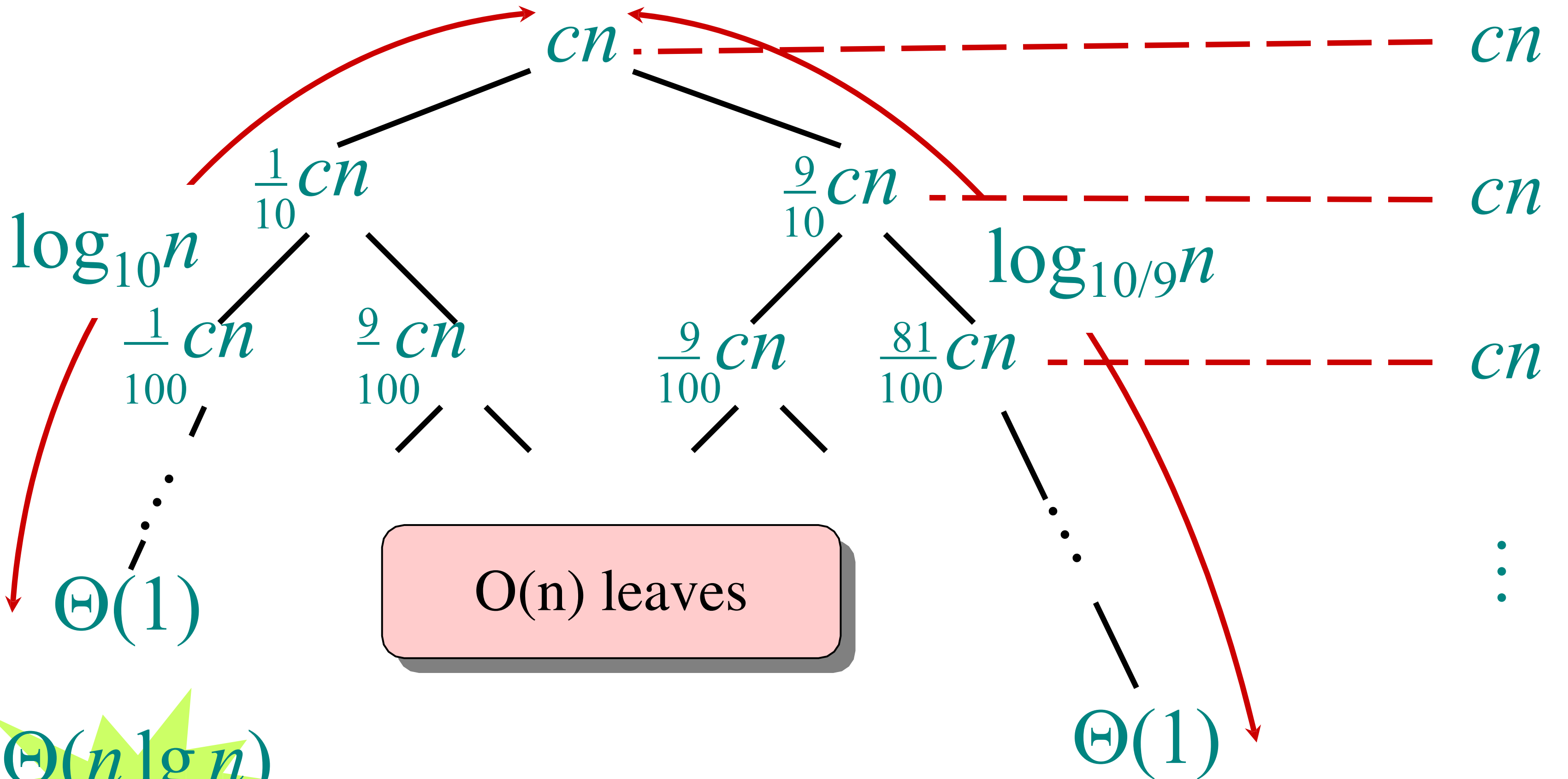
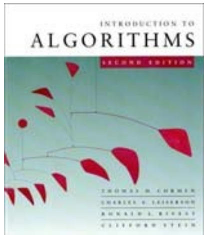


$\Theta(n \lg n)$

Lucky!

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$



More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$$L(n) = 2U(n/2) + \Theta(n) \quad \textit{lucky}$$

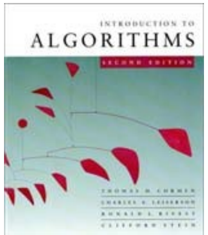
$$U(n) = L(n-1) + \Theta(n) \quad \textit{unlucky}$$

Solving:

$$
\begin{aligned}
L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\
&= 2L(n/2-1) + \Theta(n) \\
&= \Theta(n \lg n)
\end{aligned}
$$

Lucky!

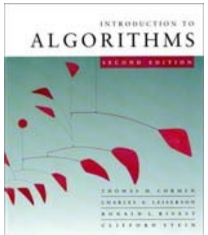
How can we make sure we are usually lucky?



Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.



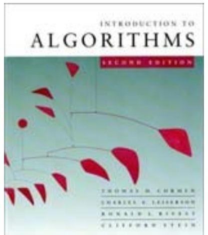
Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

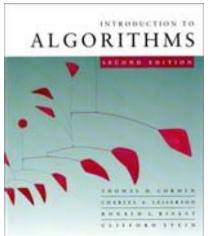
$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

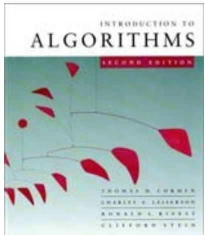
$$= \sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)$$



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

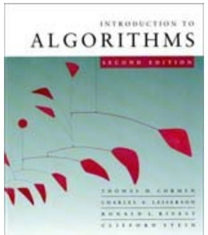


Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Linearity of expectation.



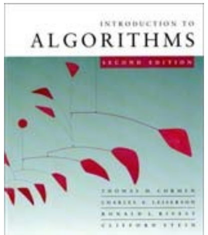
Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of X_k from other random choices.



Calculating expectation

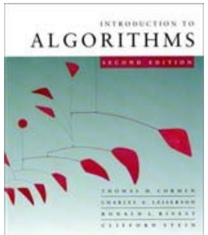
$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

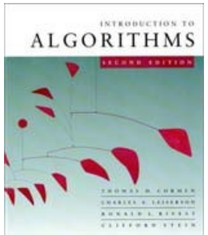
$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.



Hairy recurrence

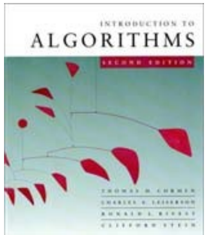
$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \leq an \lg n$ for constant $a > 0$.

- Choose a large enough so that $an \lg n$ dominates $E[T(n)]$ for sufficiently small $n \geq 2$.

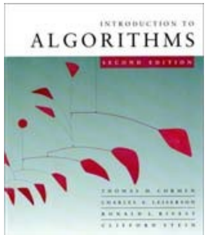
Use fact: $\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).



Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.



Substitution method

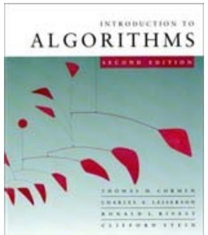
$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\leq an \lg n\,,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.