Random Walks on Graphs - Part I

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CSE, IITKGP

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Social Networks: underlying data

The underlying data is naturally a graph

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- Top k Friend recommendation to X when he joins Facebook

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- A wide variety of interesting real world applications can be framed as ranking entities in a graph
- A graph-theoretic measure for ranking nodes as well as similarity: for example, two entities are similar, if lots of short paths between them.
- Random walks have proven to be a simple, but powerful mathematical tool for extracting this information.

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- Then we select a neighbor of this node and move to it, and so on
- The (random) sequence of nodes selected this way is a *random walk* on the graph

Adjacency and Transition Matrix

$n \times n$ Adjacency matrix A

- $A(i,j)$: weight on edge from i to j
- If the graph is undirected $A(i,j) = A(j,i)$, i.e. A is symmetric

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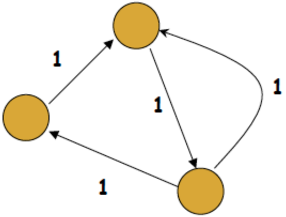
Adjacency and Transition Matrix: Example

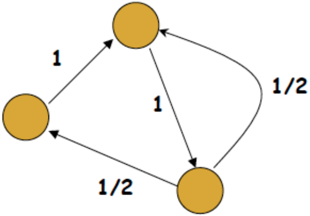
0	1	0
0	0	1
1	1	0

Adjacency matrix A

0	1	0
0	0	1
1/2	1/2	0

Transition matrix P





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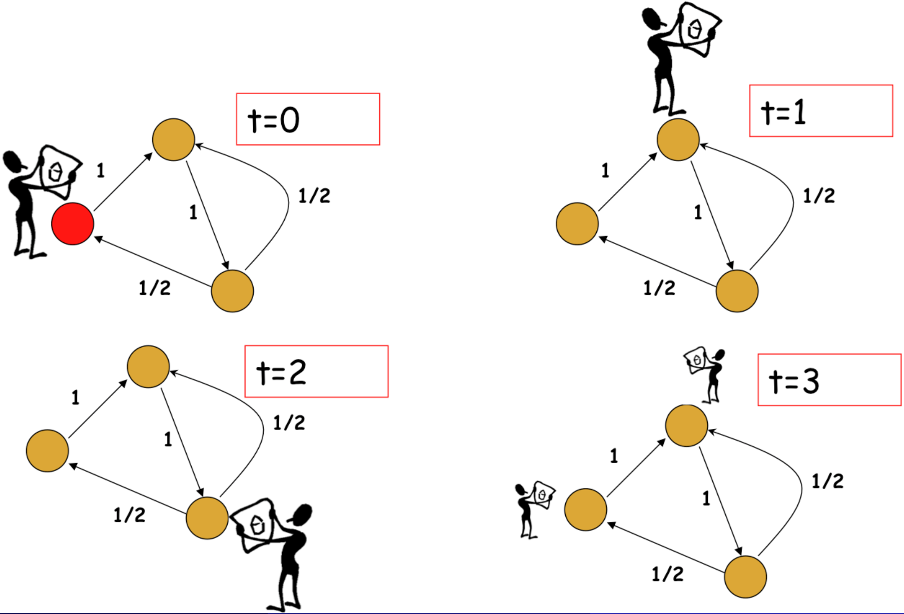

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Stationary Distribution

- When the surfer keeps walking for a long time
- When the distribution does not change anymore, i.e. x_{T+1} = x_T
- *For well-behaved graphs, this does not depend on the start distribution*

What is a stationary distribution?

- Stationary distribution at a node is related to the amount of time a random walker spends visiting that node
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- This is the left eigenvector of the transition matrix

Interesting questions

Does a stationary distribution always exist? Is it unique?

Yes, if the graph is "well-behaved"

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How fast the random surfer approach this stationary distribution?

Mixing time

Well behaved graphs

Irreducible

There is a path from every node to every other node.

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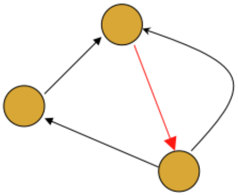
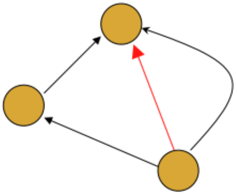


Irreducible



Not irreducible

Well behaved graphs

Aperiodic

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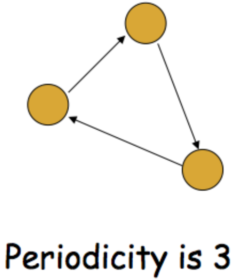

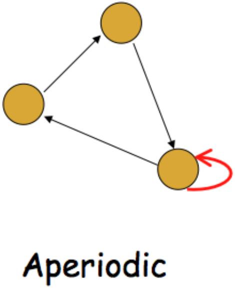

Perron Frobenius Theorem: Implications

Theorem Statement

Let $A = (a_{ij})$ be an $n \times n$ positive matrix: $a_{ij} > 0 \forall 1 \leq i,j \leq n$. Then

- There is a positive real number r, such that r is an eigenvalue of A and any other eigenvalue is strictly smaller than r in absolute value.

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Markov Chain: irreducible and aperiodic

- For any matrix A with eigenvalue σ, $|\sigma| \leq max_i \sum_j |A_{ij}|$.
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- Since P is *row stochastic*, the largest eigenvalue of the transition matrix will be equal to 1 and all other eigenvalues will be strictly less than 1
- Let the eigenvalues of P be $\{\sigma_i | i = 0 : n-1\}$ in non-decreasing order of σ_i
- $\sigma_0 = 1 > \sigma_1 \geq \sigma_2 \geq \ldots \sigma_n$

Perron Frobenius Theorem: Implications

- $v_0 = v_0 P$ (unique for a well-behaved graph)
- Let x be an arbitrary initial distribution

$$x = \sum_{i=1}^{n} a_i u_i$$

- $xP = \sum_{i=1}^{n} a_i(u_i P)$
- $= \sum_{i=1}^{n} a_i(\sigma_i u_i)$
- Similarly, $xP^k = \sum_{i=1}^{n} a_i({\sigma_i}^k u_i)$
- $xPPP \ldots P = xP^k$ tends to v_0 as k goes to infinity.

Perron Frobenius Theorem: Implications

$$xP^k = \sigma_1{}^k\{a_1u_1 + a_2\left(\frac{\sigma_2}{\sigma_1}\right)^k u_2 + \ldots + a_n\left(\frac{\sigma_n}{\sigma_1}\right)^k u_n\}$$

$u_1 = v_0$, thus xP^k approaches to v_0 as k goes to infinity with a speed in the order of σ_2/σ_1 exponentially.

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- $1_{n\times 1}$ is the right eigenvector of P with eigenvalue 1, since P is stochastic, i.e. $P^* 1_{n\times 1} = 1_{n\times 1}$
- Hence, $u_i{}^* 1_{n\times 1} = 1$ for $i = 1$, 0 otherwise (relation between left and right eigen vectors)
- Now, $1 = x^* 1_{n\times 1} = a_1 u_1{}^* 1_{n\times 1} = a_1$ (Why?)

Important Parameters of a random walk

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$Cov^+(G)$: Cover and return to start

Mixing Rate

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- How fast the random walk converges to its limiting distribution
- Mixing rate for some graphs can be very small: $O(logn)$
- Mixing rate depends on the spectral gap: $1 - \sigma_2$, where σ_2 is the second highest eigen value
- Smaller the value of σ_2, larger is the spectral gap, faster is the mixing rate

PageRank (Page and Brin, 1998)

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Basic Intuition

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- $v(i) = \sum_{j \to i} \frac{v(j)}{deg^{out}(j)}$
- v is the stationary distribution of the Markov chain
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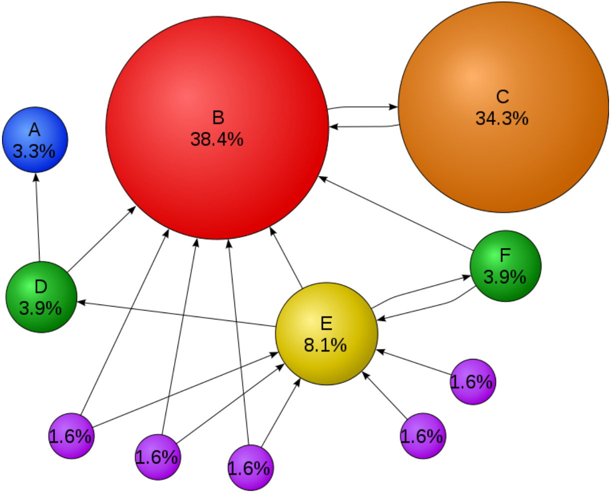
- "*The $25,000,000,000 Eigenvector: The Linear Algebra Behind Google*"

PageRank Example (Source: Wikipedia)



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$$\tilde{P} = (1-c)P + cU$$

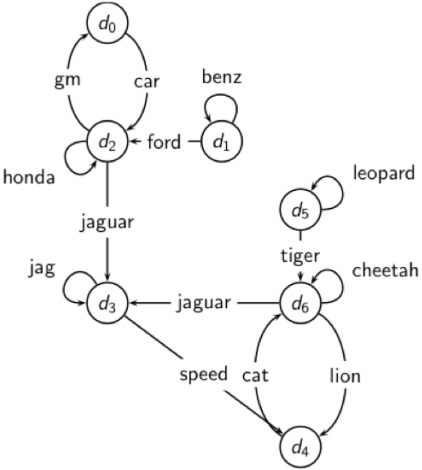
$$U_{ij} = \frac{1}{n} \forall i,j$$

Computing PageRank: The Power Method

- Start with any distribution x_0, e.g. uniform distribution
- Algorithm: multiply x_0 by increasing powers of P until convergence
- After one step, $x_1 = x_0 P$, after k steps $x_k = x_0 P^k$
- Regardless of where we start, we eventually reach the steady state v_0

PageRank: Example

Example web graph



From “Introduction to Information Retrieval” slides

PageRank: Example

Transition (probability) matrix

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0.00	0.00	1.00	0.00	0.00	0.00	0.00
d_1	0.00	0.50	0.50	0.00	0.00	0.00	0.00
d_2	0.33	0.00	0.33	0.33	0.00	0.00	0.00
d_3	0.00	0.00	0.00	0.50	0.50	0.00	0.00
d_4	0.00	0.00	0.00	0.00	0.00	0.00	1.00
d_5	0.00	0.00	0.00	0.00	0.00	0.50	0.50
d_6	0.00	0.00	0.00	0.33	0.33	0.00	0.33

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From "Introduction to Information Retrieval" slides

PageRank: Example

Transition matrix with teleporting

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0.02	0.02	0.88	0.02	0.02	0.02	0.02
d_1	0.02	0.45	0.45	0.02	0.02	0.02	0.02
d_2	0.31	0.02	0.31	0.31	0.02	0.02	0.02
d_3	0.02	0.02	0.02	0.45	0.45	0.02	0.02
d_4	0.02	0.02	0.02	0.02	0.02	0.02	0.88
d_5	0.02	0.02	0.02	0.02	0.02	0.45	0.45
d_6	0.02	0.02	0.02	0.31	0.31	0.02	0.31

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From "Introduction to Information Retrieval" slides

Power method vectors $\vec{x}P^k$

	$\vec{x}$	$\vec{x}P^{1}$	$\vec{x}P^{2}$	$\vec{x}P^{3}$	$\vec{x}P^{4}$	$\vec{x}P^{5}$	$\vec{x}P^{6}$	$\vec{x}P^{7}$	$\vec{x}P^{8}$	$\vec{x}P^{9}$	$\vec{x}P^{10}$	$\vec{x}P^{11}$	$\vec{x}P^{12}$	$\vec{x}P^{13}$
d_0	0.14	0.06	0.09	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05
d_1	0.14	0.08	0.06	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
d_2	0.14	0.25	0.18	0.17	0.15	0.14	0.13	0.12	0.12	0.12	0.12	0.11	0.11	0.11
d_3	0.14	0.16	0.23	0.24	0.24	0.24	0.24	0.25	0.25	0.25	0.25	0.25	0.25	0.25
d_4	0.14	0.12	0.16	0.19	0.19	0.20	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21
d_5	0.14	0.08	0.06	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
d_6	0.14	0.25	0.23	0.25	0.27	0.28	0.29	0.29	0.30	0.30	0.30	0.30	0.31	0.31

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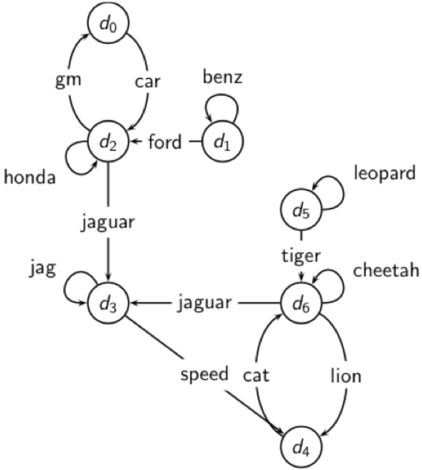

	PageRank
d_0	0.05
d_1	0.04
d_2	0.11
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d_4	0.21
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d_6	0.31

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From "Introduction to Information Retrieval" slides

Personalized PageRank

- We are looking for the vector v such that

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- r is a distribution over web-pages
- If r is the uniform distribution we get pagerank
- What happens if r is non-uniform? $\rightarrow$ Pesonalization

Personalized PageRank

- The only difference is that we use a non-uniform teleportation distribution, i.e. at any time step, teleport to a set of webpages.
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- r is a non-uniform preference vector specific to a user.
- v gives "personalized views" of the web.

Topic Sensitive PageRank

- Divide the webpages into 16 broad categories

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- For each category, compute the biased personalized pagerank vector by teleporting uniformly to websites under that category.

Topic Sensitive PageRank

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- Final pageRank vector is computed by a linear combination of the biased pagerank vectors computed offline

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- **Authorities:** pages which are good sources of information about a given topic
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- *Works on a subgraph* - can consist of top k search results for the given query from a standard text-based engine

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- A node is a good hub if it points to many good authorities, whereas a node is a good authority if many good hubs point to it.
- $a(i) \leftarrow \sum_{j:j \in I(i)} h(j)$
- $h(i) \leftarrow \sum_{j:j \in O(i)} a(j)$

Hubs and Authorities

- $a = A^T h$, $h = Aa$
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- $A^T A(i,j) = \sum_k A(k,i)A(k,j)$: number of nodes which point to both i and j, *co-citation matrix*

How to compute hub and authority scores

- Do a regular web search first
- Call the search result the root set
- Find all pages that are linked to or link to pages in the root set
- Call first larger set the base set
- Finally, compute hubs and authorities for the base set (which we'll view as a small web graph)

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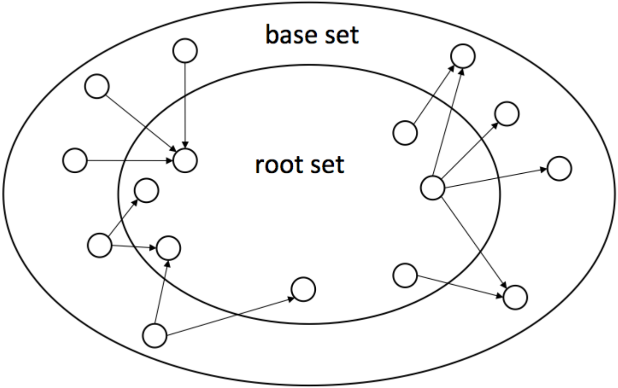
HITS: Example

Root set and base set (1)



The base set

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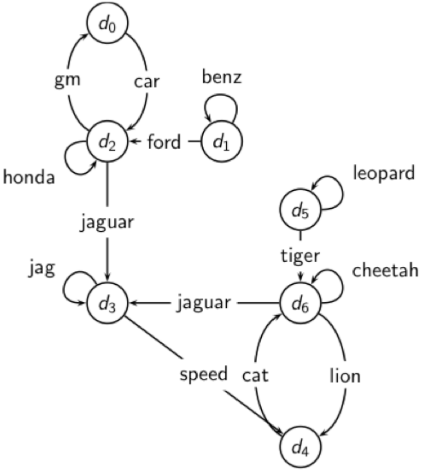
Example web graph



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HITS: Example

Raw matrix A for HITS

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0	0	1	0	0	0	0
d_1	0	1	1	0	0	0	0
d_2	1	0	1	2	0	0	0
d_3	0	0	0	1	1	0	0
d_4	0	0	0	0	0	0	1
d_5	0	0	0	0	0	1	1
d_6	0	0	0	2	1	0	1

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HITS: Example

Hub vectors h_0, $\vec{h}_i = \frac{1}{d_i} A^* a_i$, $i \geq 1$

	$\vec{h}_0$	$\vec{h}_1$	$\vec{h}_2$	$\vec{h}_3$	$\vec{h}_4$	$\vec{h}_5$
d_0	0.14	0.06	0.04	0.04	0.03	0.03
d_1	0.14	0.08	0.05	0.04	0.04	0.04
d_2	0.14	0.28	0.32	0.33	0.33	0.33
d_3	0.14	0.14	0.17	0.18	0.18	0.18
d_4	0.14	0.06	0.04	0.04	0.04	0.04
d_5	0.14	0.08	0.05	0.04	0.04	0.04
d_6	0.14	0.30	0.33	0.34	0.35	0.35

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HITS: Example

Authority vector $\vec{a} = \frac{1}{c_i} A^T * \vec{h}_{i-1}, i \geq 1$

	a_1	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_6$	$\vec{a}_7$
d_0	0.06	0.09	0.10	0.10	0.10	0.10	0.10
d_1	0.06	0.03	0.01	0.01	0.01	0.01	0.01
d_2	0.19	0.14	0.13	0.12	0.12	0.12	0.12
d_3	0.31	0.43	0.46	0.46	0.46	0.47	0.47
d_4	0.13	0.14	0.16	0.16	0.16	0.16	0.16
d_5	0.06	0.03	0.02	0.01	0.01	0.01	0.01
d_6	0.19	0.14	0.13	0.13	0.13	0.13	0.13

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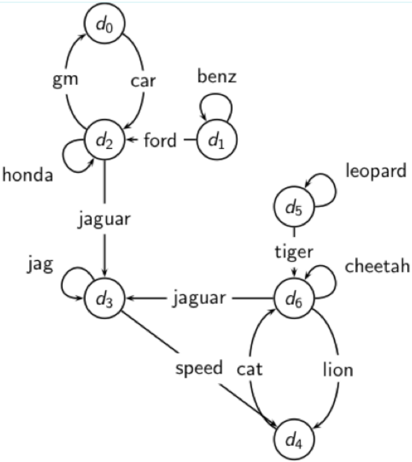
HITS: Example

Example web graph



	a	h
d_0	0.10	0.03
d_1	0.01	0.04
d_2	0.12	0.33
d_3	0.47	0.18
d_4	0.16	0.04
d_5	0.01	0.04
d_6	0.13	0.35

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