Practice Problems: Linear Programming

January 19, 2024

1. Prove that, for a feasible solution x for a primal LP and a feasible solution y for the dual, x and y are optimal solutions of their respective LPs if and only if complementary slackness holds:
   (a) For every $i \in [n]$, either $x_i = 0$ or $\sum_{j=1}^{m} a_{ji} y_j = b_i$ (or both),
   (b) For every $j \in [m]$, either $y_j = 0$ or $\sum_{i=1}^{n} a_{ij} x_i = c_i$ (or both).
2. Write dual of the linear program: $\min c^T x$ s.t. $Ax \geqslant b$ and $x \geqslant 0$. What is the relationship between the coefficient matrices of the primal and dual LPs?
3. Write the dual to the following linear program.

$$\begin{aligned} \text{minimize} \quad & x + y \\ \text{s.t.} \quad & 2x + y \leqslant 3 \\ & x + 37 \leqslant 5 \\ & x, y \geqslant 0 \end{aligned}$$

4. Design an algorithm for computing minimum-cost maximum cardinality matching in a bipartite graph.
5. Write an integer linear programming formulation of the maximum flow problem. Write its linear programming relaxation. Write its dual linear program.
6. Write an integer linear programming formulation of the $s - t$ shortest path problem. Write its linear programming relaxation. Write its dual linear program.
7. Without adding missing edges with cost infinity each, change the primal-dual algorithm discussed in the class to compute a minimum cost perfect matching in a bipartite graph to output NO if there is no perfect matching in the input graph.