Practice Problems: Basics of Fixed Parameter Tractability: Kernelization, and Bounded Search

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All the problems below are from Cygan et al [CFK+15].

1. Show that there is an FPT algorithm for a parameterized problem if and only if there is an algorithm for it running in time $O(f(k) + \text{poly}(n))$ for some computable function $f(\cdot)$.

2. Show that $(c \log n)^k = k^{\mathcal{O}(k)} + \mathcal{O}(n)$.

3. In the Maximum Satisfiability problem, we are given a CNF formula F , and a non-negative integer k, and we need to decide whether F has a truth assignment satisfying at least k clauses. Design a polynomial time kernel for this problem with at most k variables and $2k$ clauses.

4. In the Edge Clique Cover problem, we are given a graph $\mathcal{G}$ and a non-negative integer k, and the goal is to decide whether the edges of $\mathcal{G}$ can be covered by at most k cliques. Design a polynomial time kernel for this problem with number of vertices $\mathcal{O}(2^k)$.

5. Prove Hall's Theorem: Let $\mathcal{G}$ be an undirected bipartite graph with bipartition (V_1, V_2). The graph $\mathcal{G}$ has a matching saturating V_1 if and only if for all $X \subseteq V_1$, we have $|N(X)| \geqslant |X|$. Moreover, if there is no matching saturating V_1, then show that a subset $X \subseteq V_1$ with $|N(X)| < |X|$ can be computed in polynomial time.

6. Expansion Lemma: A q-star, $q \geqslant 1$, is a graph with $q+1$ vertices, one vertex of degree q, called the center, and all other vertices of degree 1 adjacent to the center. Let $\mathcal{G}$ be a bipartite graph with vertex bipartition (A, B). For a positive integer q, a set of edges $M \subseteq \mathcal{E}(\mathcal{G})$ is called by a q-expansion of A into B if

   ▷ every vertex of A is incident to exactly q edges of M;
   ▷ M saturates exactly $q|A|$ vertices in B.

   Show the following: Let $\mathcal{G}$ be a bipartite graph with bipartition (A, B). Then there is a q-expansion from A into B if and only if $|N(X)| \geqslant q|X|$ for every $X \subseteq A$. Furthermore, if there is no q-expansion from A into B, then a set $X \subseteq A$ with $|N(X)| < q|X|$ can be found in polynomial time.

7. A graph $\mathcal{G}$ is a cluster graph if every connected component of $\mathcal{G}$ is a clique. In the Cluster Editing problem, we are given as input a graph $\mathcal{G}$ and an integer k, and the objective is to check whether one can edit (add or delete) at most k edges in $\mathcal{G}$ to obtain a cluster graph. That is, we look for a set $F \subseteq \binom{V(G)}{2}$ of size at most k, such that the graph $(\mathcal{V}(\mathcal{G}), (\mathcal{E}(\mathcal{G})\ F) \cup (F\ \mathcal{E}(\mathcal{G})))$ is a cluster graph.

   (a) Show that a graph $\mathcal{G}$ is a cluster graph if and only if it does not have an induced path on three vertices (sequence of three vertices u, v, w such that $\{u, v\}$ and $\{vw\}$ are edges and $\{u, w\} \notin \mathcal{E}(\mathcal{G})$.

   (b) Show a kernel for Cluster Editing with $\mathcal{O}(k^2)$ vertices.

References

[CFK+15] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.