Problem Set 6
Martingales

1. Show that the sequence $\{S_n\}$ is a martingale in each of the following settings.
   (a) $X_1, X_2, \ldots$ are a sequence of integer valued random variables and $S_0 = 0$, $S_1 = S_0 + X_1$, $\ldots$, $S_n = X_n + S_{n-1}$ are the partial sums. Assume $\mathbf{E}[X_{n+1}|S_1, \ldots, S_n] = 0$.
   (b) $X_0, X_1, \ldots$ is a discrete time Markov chain with transition matrix $P = (p_{i,j})$ and the state space S is countable. $\psi : S \to \mathbb{R}$ is a bounded function which satisfies
   $$\sum_{j \in S} p_{i,j}\psi(j) = \psi(i),$$
   and define $S_n = \psi(X_n) \quad \forall n$.
2. Suppose that $X_1, X_2, \ldots$ are independent random variables each with mean zero and finite variance. $Y_n = \sum_{i=1}^{n} X_i$ and $S_n = Y_n^2 = \sum_{i=1}^{n}$, where σ_i is the standard deviation of X_i. Show that the sequence $\{S_n\}$ a maringale with respect to the sequence $\{X_n\}$.
3. Let $X_0, X_1, \ldots$ be a sequence of random variables and let $S_n = \sum_{i=1}^{n} X_i$ for all n. Show that if $\{S_n\}$ is a martingale with respect to $\{X_n\}$ then for all $i \neq j$, $\mathbf{E}[X_i X_j] = 0$.
4. Given a bag with r red balls and b black balls, suppose that we uniformly sample n balls from the bin without replacement. Set up an appropriate martingale and use it to show that the number of red balls in the sample is tightly concentrated around $nr/(r + b)$.
5. Let $X_1, X_2, \ldots, X_n$ be independent random variables with range $[0, 1]$ and let $S_n = \sum_{i=1}^{n} X_i$. Show that
   $$\mathbf{Pr}[|S_n - \mathbf{E}[S_n]| > \lambda] \leq 2e^{-2\lambda^2}.$$
6. Suppose there are n servers sharing a communication channel. At each time step, each server sends one message with probability $1/n$. Message transmission is successful in a time step if only one server sends in that time step. What is the expected number of time steps until all servers have sent at least one messaage?