# Practice Problems on Basic Probability, PIT, and Min Cut

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1. Let $\mathcal{X}_i, i \in [n]$ be $n$ random variables each with finite support. Then prove the following.

$$\text{var}\left(\sum_{i=1}^{n} \mathcal{X}_i\right) = \sum_{i=1}^{n} \text{var}\left(\mathcal{X}_i\right) + 2 \sum_{1 \leqslant i < j \leqslant n} \text{cov}(\mathcal{X}_i, \mathcal{X}_j)$$

   where for any two random variables $\mathcal{X}$ and $\mathcal{Y}$, we define $\text{cov}(\mathcal{X}, \mathcal{Y}) = \mathbb{E}[\mathcal{X}\mathcal{Y}] - \mathbb{E}[\mathcal{X}]\mathbb{E}[\mathcal{Y}]$.

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two independent random variables. Then prove that $\mathbb{E}[\mathcal{X}\mathcal{Y}] = \mathbb{E}[\mathcal{X}]\mathbb{E}[\mathcal{Y}]$. From this conclude that, for $n$ pairwise random variables $\mathcal{X}_i, i \in [n]$, we have the following.

$$\text{var}\left(\sum_{i=1}^{n} \mathcal{X}_i\right) = \sum_{i=1}^{n} \text{var}(\mathcal{X}_i)$$

3. Compute the running time of the polynomial identity testing algorithm discussed in the class where the input polynomial is over a finite field $\mathbb{F}$ and its total degree is $d < |\mathbb{F}|$.

4. Show that the number of min cuts in every unweighted undirected graph is at most $\binom{n}{2}$.

5. Genaralize Karger's and Karger-Stein algorithms for min cut to edge weighted graphs. Assume that the weight of every edge is a positive integer.

6. In the $k$-cut problem, the input is a unweighted graph and the goal is to compute the minimum number of edges that needs to be removed to partition the graph into $k$ components. Adapt the Karger's min-cut algorithm to design a randomized $\mathcal{O}(n^{2k})$-time algorithm for the $k$-cut problem.