Practice Problems on Markov Chain and Monte Carlo Method

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1. Suppose we have a Monte Carlo randomized algorithm $\mathcal{A}$ for some decision problem Π which, on every input x, outputs correct answer with probability at least $3/4$. Suppose $\mathcal{A}$ uses $\mathcal{O}(\log n)$ random bits where n is the size of input. Prove that there exists a deterministic polynomial time algorithm for the problem Π.

2. The metropolis algorithm discussed in the class assumes that the given probability distribution assigns non-zero probability to every vertex of the Markov chain. Suitably modify the metropolis algorithm so that we do not need to make this assumption.

3. Let $\mathcal{P}$ and $\mathcal{Q}$ be two distribution over the σ-algebra $([n], 2^{[n]})$. Then prove the following.

$$2d_{TV}(\mathcal{P}, \mathcal{Q}) = ||\mathcal{P} - \mathcal{Q}||_1$$

4. Consider drunken walk on integers 0 to n where the transition probability from k to $k+1$ is $\frac{1}{3}$ for every $k \in [n-1]$, the transition probability from k to $k-1$ is $\frac{2}{3}$ for every $k \in [n]$, the transition probability from 0 to 1 is 1, and n is an absorbing state. Prove that there exists a constant $c > 1$ such that the expected number of steps to reach n from 0 is $\Omega(c^n)$.

5. A random walk (without self-loop) on a connected undirected graph is aperiodic if and only if the graph is not bipartite.

6. The Monte Carlo method for estimating the value of π discussed in the class assumes that we can draw uniform samples from a 2×2 square in $\mathbb{R}^2$ which is an uncountably infinite set. Design a (ε, δ)-approximator of π which needs to draw uniform sample from finite sets only.

7. Show that the mixing time of a random walk on an n dimensional hypercube is at most $n \ln n + n \ln(1/\varepsilon)$.

8. Consider the following card shuffling: insert the top card at a uniformly random position from 1 to n. Model the shuffling process as a Markov chain. Show that the uniform distribution is the unique stationary distribution. Show that the mixing time is at most $n \ln n + n \ln(1/\varepsilon)$.