	INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR	**Stamp / Signature of the Invigilator**

EXAMINATION (Mid Semester)	**SEMESTER (Autumn)**

Roll Number									**Section**		**Name**	
Subject Number	C	S	6	0	0	2	9	**Subject Name**	*Randomized Algorithm Design*			
Department / Center of the Student									**Additional sheets**			

Important Instructions and Guidelines for Students

1. You must occupy your seat as per the Examination Schedule/Sitting Plan.
2. Do not keep mobile phones or any similar electronic gadgets with you even in the switched off mode.
3. Loose papers, class notes, books or any such materials must not be in your possession, even if they are irrelevant to the subject you are taking examination.
4. Data book, codes, graph papers, relevant standard tables/charts or any other materials are allowed only when instructed by the paper-setter.
5. Use of instrument box, pencil box and non-programmable calculator is allowed during the examination. However, exchange of these items or any other papers (including question papers) is not permitted.
6. Write on both sides of the answer script and do not tear off any page. **Use last page(s) of the answer script for rough work.** Report to the invigilator if the answer script has torn or distorted page(s).
7. It is your responsibility to ensure that you have signed the Attendance Sheet. Keep your Admit Card/Identity Card on the desk for checking by the invigilator.
8. You may leave the examination hall for wash room or for drinking water for a very short period. Record your absence from the Examination Hall in the register provided. Smoking and the consumption of any kind of beverages are strictly prohibited inside the Examination Hall.
9. Do not leave the Examination Hall without submitting your answer script to the invigilator. **In any case, you are not allowed to take away the answer script with you.** After the completion of the examination, do not leave the seat until the invigilators collect all the answer scripts.
10. During the examination, either inside or outside the Examination Hall, gathering information from any kind of sources or exchanging information with others or any such attempt will be treated as **'unfair means'**. Do not adopt unfair means and do not indulge in unseemly behavior.

Violation of any of the above instructions may lead to severe punishment.

Signature of the Student

To be filled in by the examiner

Question Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Marks Obtained											

Marks obtained (in words)	**Signature of the Examiner**	**Signature of the Scrutineer**

Answer any FIVE questions.

1. Consider the following card shuffling: insert the top card at a uniformly random position from 1 to n. Model the shuffling process as a Markov chain.

   (a) (4 points) Show that the uniform distribution is the unique stationary distribution.

   (b) (8 points) Show that the mixing time is at most $n \ln n + n \ln(\frac{1}{\varepsilon})$.

2. (12 points) Consider two random walks on a cycle of length $2n$ starting at diagonally opposite vertices. In each random walk, we move to one of its neighbors with probability $\frac{1}{4}$ and stay at the current vertex with probability $\frac{1}{2}$. Compute the expected number of steps for them to meet.

3. (12 points) Assuming that we can pick a uniformly random clique from the set of all cliques in any graph, design a Monte Carlo algorithm to estimate the number of cliques in a graph. A clique is a subset K of vertices where we have an edge between every pair of vertices in K.

4. (12 points) Let $X_1, X_2, \ldots, X_n$ be independent random variables such that

$$\Pr[X_i = 1 - p_i] = p_i \text{ and } \Pr[X_i = -p_i] = 1 - p_i,$$

where $0 < p_i < 1$ for all $i \in [n]$. Let $X = \sum_{i=1}^n X_i$. Prove that

$$\Pr[|X| \geq a] \leq 2e^{-2a^2/n}.$$

Hint: Use the inequality $p_i e^{t(1-p_i)} + (1 - p_i)e^{-tp_i} \leq e^{t^2/8}$.

5. Consider n balls thrown randomly into n bins. Let $X_i = 1$ if i-th bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Let Y_i, $i \in [n]$ be Bernoulli random variables that are 1 with probability $p = \left(1 - \frac{1}{n}\right)^n$. Let $Y = \sum_{i=1}^{n} Y_i$.

   (a) (2 points) Show that $\mathbb{E}[X_1 X_2 \cdots X_k] \leq \mathbb{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$. (Use the fact that $(1 - k/n) \leq (1 - 1/n)^{kn}$ for all positive integers n, k.)

   (b) (6 points) Prove that $\mathbb{E}\left[e^{tX}\right] \leq \mathbb{E}\left[e^{tY}\right]$ for all $t \geq 0$. (Use the expansion for e^{tX}, e^{tY}; compare $\mathbb{E}\left[X^k\right]$ and $\mathbb{E}\left[Y^k\right]$).

   (c) (4 points) Derive the following Chernoff bound:

$$\Pr[X \geq (1+\delta)\mathbb{E}[X]] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mathbb{E}[Y]}.$$

6. (12 points) Let $\mathbf{A} \in \{0,1\}^{n\times m}$ and $\vec{b} \in \{0,1\}^n$ be uniformly and independently distributed. Define for $\vec{x} \in \{0,1\}^m$, a random variable $Z_{\vec{x}} = (\mathbf{A}\vec{x} + \vec{b}) \bmod 2$. Show that the random variables $\{Z_{\vec{x}} \,|\, \vec{x} \in \{0,1\}^m\}$ are pairwise independent.

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