1. (a) n balls $\rightarrow$ n bins

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ bin is empty} \\ 0 & \text{otherwise} \end{cases}$$

Let $\underbrace{X}_{\text{no. of empty bins}} = \sum_{i=1}^{n} X_i$

$Pr[X_i = 1] = \left(1 - \frac{1}{n}\right)^n$

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} E[X_i] \\ &= n \, Pr[X_i = 1] \\ &= n \underbrace{\left(1 - \frac{1}{n}\right)^n}_{\approx 1/e} \approx \frac{n}{e} \end{aligned}$$

(b) m balls $\rightarrow$ n bins

Define X_i & X as in (a).

$$Pr[X_i = 1] = \left(1 - \frac{1}{n}\right)^m$$

$$E[X] = n\left(1 - \frac{1}{n}\right)^m = n\left(1 - \frac{1}{n}\right)^{n \cdot \frac{m}{n}} \approx \frac{n}{e^{m/n}}$$

2. (a) Let $X - \mu = Y$.

$$E[Y] = E[X] - \quad = 0$$

$$\mathrm{Var}[Y] = E[(Y - E[Y])^2] = E[(X-\mu)^2] = \sigma^2$$

$$\Pr[X - \mu \geq s] = \Pr[Y \geq s]$$

$$= \Pr[Y + a \geq s + a]$$

$$\leq \Pr[(Y+a)^2 \geq (s+a)^2]$$

$$\leq \frac{E[(Y+a)^2]}{(s+a)^2}$$

$$= \frac{E[Y^2] + a^2}{(s+a)^2} \quad \left(\text{since } E[Y] = 0\right)$$

$$= \frac{\sigma^2 + a^2}{(s+a)^2} \quad \left(\text{minimized for } a = \frac{\sigma^2}{s}\right)$$

$$= \frac{\sigma^2(s^2 + \sigma^2)}{(s^2+\sigma^2)^2} \quad \left(\text{substituting } a = \frac{\sigma^2}{s}\right)$$

$$= \frac{\sigma^2}{s^2 + \sigma^2}$$

$$\Pr[X - \mu \geq t\sigma] \leq \frac{\sigma^2}{t^2\sigma^2 + \sigma^2} = \frac{1}{1+t^2}$$

(b) $\Pr[|X-\mu| \geq t\sigma]$

$= \Pr[X-\mu \geq t\sigma] + \Pr[X-\mu \leq -t\sigma]$

$\leq \dfrac{2}{1+t^2}$

(c) The inequality in part (a) is better than Chebyshev's inequality as

$\dfrac{1}{1+t^2} < \dfrac{1}{t^2}$

But for 2-sided tail bounds, Chebyshev's inequality is always better than part (b) unless $t < 1$.

discussed in class

3. (a) $\Pr[X=0] \leq \Pr[|X - E[X]| \geq E[X]]$

Chebyshev's $\leq \dfrac{Var[X]}{E[X]^2}$

defn. of variance $= \dfrac{E[X^2] - E[X]^2}{E[X]^2}$

(b) $\Pr[X \neq 0] = \Pr[X \geq 1]$ — X is non-negative

Markov's $\leq E[X]$

Cauchy-Schwartz Inequality

X, Y: $E[XY]^2 \leq E[X^2]E[Y^2]$

Proof: Assume (w.l.g) X, Y are non-negative r.v.s
Also assume $E[X^2]E[Y^2] > 0$ (for o.w, at least one of X, Y is 0 with prob. 1)

X', Y': non-negative r.v.s s.t. $E[X'^2] = E[Y'^2] = 1$
Then, $0 \leq E[(X'-Y')^2] = 2 - 2E[X'Y']$
i.e., $E[X'Y'] \leq 1$ — (*)

Let $X' = \dfrac{X}{\sqrt{E[X^2]}}$ & $Y' = \dfrac{Y}{\sqrt{E[Y^2]}}$

From (*), we have $E[X'Y']^2 = \dfrac{E[XY]^2}{E[X^2]E[Y^2]} \leq 1$

from which the result follows.

Let $Y = \begin{cases} 1 & \text{if } X \geq 1 \\ 0 & \text{o.w} \end{cases}$

$\Pr[X \neq 0] = \Pr[X \geq 1] = E[Y]$

By Cauchy-Schwartz inequality,

$$E[Y]\,E[X^2] \geq E[YX]^2 = E[X]^2$$

i.e., $$E[Y] = \Pr[X \geq 1] \geq \frac{E[X]^2}{E[X^2]}$$

7. (a) Discussed in class

(b) Consider the i^{th} row $\vec{a}_i^T = (a_{i1}, a_{i2}, \ldots, a_{im})$ of A.

Let k denote the no. of 1's in $\vec{a}_i$.

<u>Case 1</u>: $k \leq \sqrt{4m \ln n}$

Then irrespective of b,

$$|\vec{a}_i^T \vec{b}| = |c_i| \leq \sqrt{4m \ln n}$$

<u>Case 2</u>: $k > \sqrt{4m \ln n}$

Then there are k non-zero terms in

$$C_i = \sum_{j=1}^{m} a_{ij} b_j$$

All the non-zero terms are independent & each term is 1 or -1 with probability $\frac{1}{2}$.

Using part(a), we have

$$\Pr\left(|C_i| > \sqrt{4m\ln n}\right) \le 2e^{-\frac{4m\ln n}{2k}}$$
$$\le \frac{2}{n^2} \quad \left(\text{since } k \le m\right)$$

$$\Pr\left[|A\vec{b}|_\infty > \sqrt{4m\ln n}\right]$$
$$= \Pr\left[\bigcup_{i=1}^{n}\left(|C_i| > \sqrt{4m\ln n}\right)\right]$$
$$\le n \cdot \frac{2}{n^2} = \frac{2}{n}.$$

8. (a) Assume n is even.

Write $x \in \{0,1\}^n$ as (x^1, x^2)

where $x^1, x^2 \in \{0,1\}^{n/2}$

Let π be a permutation s.t.

$$\pi(x^1, 0^{n/2}) = (0^{n/2}, x^1) \text{ for all } x^1 \in \{0,1\}^{n/2}.$$

Length of the route from $(x^1, 0)$ to $(0^{n/2}, x^1) = n$

Number of possible choices for $x^1 = 2^{n/2}$

All the $2^{n/2}$ paths pass through $(0^{n/2}, 0^{n/2})$.

There are only n outgoing edges from $(0^{n/2}, 0^{n/2})$.

$\therefore$ at least $2^{n/2}/n$ time steps required to send all the packets from vertices $(x', 0^{n/2})$.

(b) Refer to the text-book (Motwani-Raghavan) for a detailed discussion.

6. [Only a hint. If you cannot solve it before mid-sem, I will share the solution]

Y: no. of rounds before every bin is non-empty.

In the coupon collector's problem, we do not care whether a bin is empty or non-empty & in each round only 1 player throws a ball in a bin.

$E[Y]$ is certainly much smaller compared to the expected no. of rounds in the coupon collector's problem which is $O(n \log n)$

Given that n balls are thrown instead of 1 $E[Y] = O(\log n)$

It turns out that $E[Y]$ has a better upper bound.

[In what follows all logarithms are to base 2.]

Define a function $f: \mathbb{N} \to \mathbb{N}$ as

$$f(0) = 1$$
$$f(i+1) = 2^{f(i)} \quad \text{for } i \geq 1$$

For a fixed n, the smallest $k \in \mathbb{N}$ for which $\frac{n}{f(k)} < 0$ is the no. of time we take logarithm of n repeatedly for the result to be 0. That is

$$k = \log^* n = \begin{cases} 1 + \log^*(\log n) & \text{if } n > 1 \\ 0 & \text{if } n \leq 1 \end{cases}$$

Now show the following:

After round i and before round $i+1$, if μ_i is the expected no. of empty bins then,

$$\frac{n}{f(i+1)} \leq \mu_i < \frac{n}{f(i)}$$

From this it follows that after $k = \log^* n$ rounds $\mu_k < 0$ i.e., all bins are non-empty.