Indian Institute of Technology Kharagpur

CS31005 Algorithms II – Class Test 2 Total marks: 30 Duration: 1 hr

Name: ______________________________

Roll Number: ______________________________

Answer all questions.

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1. (10 points) Consider a variant $\mathcal{A}$ of the Karger Stein Algorithm:
   Input: G
   If $n \geqslant 6$, do 2 runs of the following and output the best of the 2:
   Step 1. Use Karger's algorithm to contract vertices until $\frac{n}{\sqrt{2}} + 1$ vertices are left, in the resultant graph G'
   Step 2. Recurse $\mathcal{A}$ on G'.
   If $n < 6$, run Karger's algorithm.
   Answer the following questions:

   1. Write a recurrence relation for the running time function and derive an upper bound on the running time of 1 run of the algorithm.
   2. Write a recurrence relation for the success probability of the algorithm and derive a lower bound on the success probability of 1 round of the algorithm.
   3. Analyse the running time of (possibly multiple runs of) this algorithm if we need the success probability to be a constant.

2. (a) (2 points) Define the 3-Dimensional Matching and Set Cover problems.
   (b) (2 points) Prove that both the 3-Dimensional Matching and Set Cover problems are in NP.
   (c) (6 points) Show a many-to-one Karp reduction from the 3-Dimensional Matching to Set Cover.

1. Answer to Question 1 Part 1 (6 marks): Recurrence for running time is $T(n) \leqslant 2T(\frac{n}{\sqrt{2}}) + \mathcal{O}(n^2)$.
   Base case is when $n \leqslant 6$ as $\frac{6}{\sqrt{2}} \leqslant 2$. When only 2 vertices are left, Karger's Algorithm returns all edges remaining as the minimum cut in $\mathcal{O}(m) = \mathcal{O}(n^2)$ time.
   By Master's Theorem, $T(n) = \mathcal{O}(n^2 \log n)$.

2. Answer to Question 1 Part 2 (2 marks): By choice of contracting till $\frac{n}{\sqrt{2}} + 1$ vertices, the probability of success to contract from n vertices to $\frac{n}{\sqrt{2}} + 1$ vertices is at least $1/2$.
   Recurrence for probability of success if $P(n) \geqslant 1 - (1 - \frac{1}{2}P(\frac{n}{\sqrt{2}}))^2$.
   Use induction method to show that $P(n) = c\frac{1}{\log n}$.

3. Answer to Question 1 Part 3 (2 marks): For the success probability to be at least a constant, we need to have $O(\log n)$ runs of the algorithm and report the minimum cut amongst the results from each of the runs, and therefore the total running time shall be $O(n^2 \log^2 n)$.

4. Answer to Question 2 part 3: Given an instance $(X \uplus Y \uplus Z, \mathcal{F})$ of 3D Matching (size of each set X, Y, Z is n) we construct the instance $(X \cup Y \cup Z, \mathcal{F}, n)$ for Set Cover. It needs to be checked that the universe $X \cup Y \cup Z$ can be covered with at most n sets of $\mathcal{F}$ if and only if $X \uplus Y \uplus Z$ has a 3D matching in $\mathcal{F}$.
   First suppose F' is a 3D matching, then F' is also a set cover of $X \cup Y \cup Z$ with n sets.
   On the other hand, suppose F' is a subfamily of at most n covering all elements of $X \cup Y \cup Z$. Since X, Y, Z are three disjoint sets and each set in $\mathcal{F}$ can cover exactly 3 items, we need at least n sets in F' to cover all elements. Thus, F' must have exactly n elements and we can only afford to cover each of the $3n$ universe elements exactly once - hence F' is a 3D matching for $X \uplus Y \uplus Z$.