Lecture Notes: Hall's Theorem and König's Theorem from Max-Flow Min-Cut Theorem

Palash Dey
palash.dey@cse.iitkgp.ac.in
Indian Institute of Technology Kharagpur

August 15, 2023

Theorem 1 (Max-Flow Min-Cut) *The value of a maximum* $s-t$ *flow in a network is the capacity of a minimum* $s-t$ *cut in that network.*

We prove Hall's Theorem and König's Theorem from the above max-flow min-cut Theorem.

Let $\mathcal{G}(\mathcal{V} = \mathcal{A} \biguplus \mathcal{B}, \mathcal{E})$ be a bipartite graph. We are interested in finding a perfect matching in $\mathcal{G}$— a set $\mathcal{M}$ of edges with no two having a common end point and every vertex is covered by $\mathcal{M}$. For a subset $\mathcal{X} \subseteq \mathcal{A}$, we denote its neighborhood $\{b \in \mathcal{B} : \exists x \in \mathcal{X}$ such that $\{x, b\} \in \mathcal{E}\}$ by $\mathcal{N}(\mathcal{X})$. Clearly a necessary condition for existence of a perfect matching in $\mathcal{G}$ is $|\mathcal{X}| \leqslant |\mathcal{N}(\mathcal{X})|$ for every $\mathcal{X} \subseteq \mathcal{A}$. Hall's Theorem says that this condition is also sufficient.

Theorem 2 (Hall's Theorem) *A bipartite graph* $\mathcal{G}(\mathcal{V} = \mathcal{A} \biguplus \mathcal{B}, \mathcal{E})$ *has a perfect matching if and only if* $|\mathcal{X}| \leqslant |\mathcal{N}(\mathcal{X})|$ *for every* $\mathcal{X} \subseteq \mathcal{A}$.

We orient the edges of $\mathcal{G}$ from $\mathcal{A}$ to $\mathcal{B}$, add two vertices namely s and t, add edges from ss to every vertex in $\mathcal{A}$, add edges from every vertex in $\mathcal{B}$ to t, and define the capacity of every edge to be one. Let us call the resulting graph $\mathcal{G}'$. It is easy to see that the value of a maximum flow from s to t is k if and only if the size maximum matching in $\mathcal{G}$ is k. We know from max-flow min-cut Theorem that there is an $s-t$ cut $(\mathcal{U}, \mathcal{V} \setminus \mathcal{U})$ of capacity (which is the number of edges in this case since all the capacities are one) k. Let us define $\mathcal{A}_1 = \mathcal{U} \cap \mathcal{A}$ and $\mathcal{B}_1 = \mathcal{U} \cap \mathcal{B}$. Let the number of edges from $\mathcal{A}_1$ to $\mathcal{B} \setminus \mathcal{B}_1$ be ℓ and $n = |\mathcal{A}| = |\mathcal{B}|$. Then we have the following.

$$|\mathcal{A} \setminus \mathcal{A}_1| + |\mathcal{B}_1| + \ell = k$$
$$\Rightarrow |\mathcal{B}_1| + \ell = k - |\mathcal{A} \setminus \mathcal{A}_1|$$

We observe

$$|N(\mathcal{A}_1)| \leqslant |\mathcal{B}_1| + \ell = k - |\mathcal{A} \setminus \mathcal{A}_1| = |\mathcal{A}_1| - (n-k)$$

which proves Hall's Theorem.

A vertex cover of a graph is a set of vertices which contains at least one end point of every edge. Hence, the size of a maximum matching puts a lower bound on the size of the minimum vertex cover of the graph. König's Theorem says that the lower bound is always achievable in bipartite graphs.

Theorem 3 (König's Theorem) *The size of a minimum vertex cover is the same as the size of a maximum cardinality matching in every bipartite graph.*

Continuing our set-up from the proof of Hall's Theorem, we consider the set $\mathcal{W} = \mathcal{N}(\mathcal{A}_1) \cup (\mathcal{A} \setminus \mathcal{A}_1)$ which is a vertex cover of $\mathcal{G}$. Now

$$|\mathcal{W}| = |\mathcal{N}(\mathcal{A}_1)| + |\mathcal{A} \setminus \mathcal{A}_1| \leqslant |\mathcal{A}_1| - n + k + (n - |\mathcal{A}_1|) = k$$

which proves König's Theorem (recall, k is the size of a maximum cardinality matching in $\mathcal{G}$).