Date of Examination: 26$^{\text{th}}$ March 2022
Duration: 30 minutes + 5 minutes (for scanning, concatenating, and uploading)
Full Marks: 20
Subject: CS21003 Algorithms I

Part I

1. A tree is called a ternary tree if every node has at most 3 children. Let us call a ternary tree KGP tree if, at every node x, the difference between the height of all the subtrees rooted at the children of x differ by at most 5. Find the asymptotic behavior of the maximum height of any KGP tree with n nodes.

   [10 **Marks**]

   Solution sketch. Let $f(n)$ be the minimum number of nodes in any KGP tree of height n. Then we have the following.

   $$f(n) = \begin{cases} f(n-1) + 2f(n-5) & \text{if } n \geqslant 5 \\ n & \text{otherwise} \end{cases}$$

   Solving the above recurrence relation, we obtain $f(n) \geqslant 1.01^n$. Hence, the maximum height of any KGP tree with n nodes is $\mathcal{O}(\log n)$. $\square$

2. Show that any comparison-based algorithm for finding the median of an integer array of size n must use at least $\Omega(n)$ comparisons.

   [10 **Marks**]

   Solution sketch. The algorithm must make $\frac{n}{6}$ comparisons. Otherwise, the algorithm does not have any information about at least $\frac{2n}{3}$ elements. Any of these $\frac{2n}{3}$ elements can be made median thereby contradicting the correctness of the algorithm. $\square$

All the best