Algorithmic Game Theory
Practice Problems: Mechanism Design, Stable Matching

Palash Dey
Indian Institute of Technology, Kharagpur

October 26, 2021

1. (Inspired by an exercise from [Nar14]) Consider a scenario with a set [5] five sellers selling identical items with valuations $v_1 = 23, v_2 = 15, v_3 = 11, v_4 = 8, v_5 = 2$ and one buyer. Compute VCG payments in each of the following cases.

   (i) The buyer wishes to buy 3 items and each seller can supply at most one items.

   (ii) The buyer wishes to buy 3 items and each seller can sell at most 2 items.

   (iii) The buyer wishes to buy 6 items and each seller can sell at most 2 items.

2. Prove that the second price auction is the only auction for selling single item to one of n potential buyers that is DSIC, sells the item to a buyer having highest valuation, and losers pay nothing.

3. Groves' payment rule implements any allocatively efficient allocation rule in dominant strategy equilibrium. In general, Groves' payment rules does not make the resulting social choice function strictly budget balanced. Actually there is no payment rule which can implement an allocatively efficient allocation rule in DSIC and be SBB since it would violate an impossibility result by Green and Laffont. We now see another payment rule, known as dAGVA rule, in the class of Groves' payment rules which is defined as follows. Let k^* be any AE rule.

$$\xi_i(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[\sum_{j \neq i} \nu_j(k^*(\theta_i, \theta_{-i}), \theta_j)\right], \ \forall i \in [n]$$

$$h_i(\theta_{-i}) = -\frac{1}{n-1}\sum_{j \neq i} \xi_j(\theta_j)$$

   Show the following.

   (i) The social choice function induced by the dAGVA payment rule is strictly budget balanced.

   (ii) The social choice function induced by the dAGVA payment rule is Bayesian incentive compatible (BIC).

4. Consider a stable matching instance with a set $\mathcal{A}$ of n men and another set $\mathcal{B}$ of n women. For each woman $w \in \mathcal{B}$, we define $h(w)$ to be the least preferred man $m \in \mathcal{A}$ by the woman w with whom she can be matched in some stable matching. A matching is called women-pessimal if every woman $w \in \mathcal{B}$ is matched with $h(w)$. Prove that the stable matching output by the men-proposal deferred acceptance algorithm is women-pessimal.

5. In a stable matching instance with sets $\mathcal{A}$ and $\mathcal{B}$ of n men and women, suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ be two stable matchings. Define $\mathcal{M}_3 = \{(a, b) \in \mathcal{A} \times \mathcal{B} : \mathcal{M}_1(a) = \mathcal{M}_2(a) = b \text{ or } b = \mathcal{M}_1(a) \succ_a \mathcal{M}_2(a) \text{ or } b = \mathcal{M}_2(a) \succ_a \mathcal{M}_1(a)\}$; that is, in $\mathcal{M}_3$, every man $a \in \mathcal{A}$ gets his better partner between $\mathcal{M}_1(a)$ and $\mathcal{M}_2(a)$. Prove that $\mathcal{M}_3$ is also a stable matching.

References

[Nar14] Y. Narahari. *Game Theory and Mechanism Design*. World Scientific Publishing Company Pte. Limited, 2014.