Number Systems

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Number Representation

Binary

Hexadecimal

Decimal

Topics to be Discussed

- *How are numeric data items actually stored in computer memory?*
- *How much space (memory locations) is allocated for each type of data?*
  - *int, float, char, double, etc.*
- *How are characters and strings stored in memory?*
  - *Already discussed.*

Number System :: The Basics

- *We are accustomed to using the so-called **decimal number system**.*
  - *Ten digits :: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9*
  - *Every digit position has a weight which is a power of 10.*
  - *Base or radix is 10.*

- *Example:*

$$234 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$$

$$250.67 = 2 \times 10^2 + 5 \times 10^1 + 0 \times 10^0 + 6 \times 10^{-1} + 7 \times 10^{-2}$$

Binary Number System

- **Two digits:**
  - 0 and 1.
  - Every digit position has a weight which is a power of 2.
  - Base or radix is 2.

- **Example:**

$$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

$$101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$$

Binary-to-Decimal Conversion

- *Each digit position of a binary number has a weight.*
  - *Some power of 2.*

- *A binary number:*

$$B = b_{n-1}\, b_{n-2} \,.....b_1\, b_0 \,.\, b_{-1}\, b_{-2} \,.....\, b_{-m}$$

Corresponding value in decimal:

$$D = \sum_{i=-m}^{n-1} b_i\, 2^i$$

Examples

1. 101011 → $1x2^5 + 0x2^4 + 1x2^3 + 0x2^2 + 1x2^1 + 1x2^0$

   $= 43$

   $(101011)_2 = (43)_{10}$

1. .0101 → $0x2^{-1} + 1x2^{-2} + 0x2^{-3} + 1x2^{-4}$

   $= .3125$

   $(.0101)_2 = (.3125)_{10}$

1. 101.11 → $1x2^2 + 0x2^1 + 1x2^0 + 1x2^{-1} + 1x2^{-2}$

   5.75

   $(101.11)_2 = (5.75)_{10}$

Decimal-to-Binary Conversion

- Consider the integer and fractional parts separately.
- For the integer part,
  - Repeatedly divide the given number by 2, go on accumulating the remainders, until the number becomes zero.
  - Arrange the remainders in reverse order.
- For the fractional part,
  - Repeatedly multiply the given fraction by 2.
    - Accumulate the integer part (0 or 1).
    - If the integer part is 1, chop it off.
  - Arrange the integer parts in the order they are obtained.

Example 1 :: 239

2	239	
2	119	--- 1
2	59	--- 1
2	29	--- 1
2	14	--- 1
2	7	--- 0
2	3	--- 1
2	1	--- 1
2	0	--- 1

$(239)_{10} = (11101111)_2$

Example 2 :: 64

2	64	
2	32	--- 0
2	16	--- 0
2	8	--- 0
2	4	--- 0
2	2	--- 0
2	1	--- 0
2	0	--- 1

$(64)_{10} = (1000000)_2$

Example 3 :: .634

$.634 \times 2 = 1.268$

$.268 \times 2 = 0.536$

$.536 \times 2 = 1.072$

$.072 \times 2 = 0.144$

$.144 \times 2 = 0.288$

:

:

$(.634)_{10} = (.10100......)_2$

Example 4 :: 37.0625

$(37)_{10} = (100101)_2$

$(.0625)_{10} = (.0001)_2$

$\therefore (37.0625)_{10} = (100101\,.\,0001)_2$

Hexadecimal Number System

- *A compact way of representing binary numbers.*
- *16 different symbols (radix = 16).*

0	→	0000	8	→	1000
1	→	0001	9	→	1001
2	→	0010	A	→	1010
3	→	0011	B	→	1011
4	→	0100	C	→	1100
5	→	0101	D	→	1101
6	→	0110	E	→	1110
7	→	0111	F	→	1111

Binary-to-Hexadecimal Conversion

- For the integer part,
  - Scan the binary number from right to left.
  - Translate each group of four bits into the corresponding hexadecimal digit.
    - Add leading zeros if necessary.
- For the fractional part,
  - Scan the binary number from left to right.
  - Translate each group of four bits into the corresponding hexadecimal digit.
    - Add trailing zeros if necessary.

Examples

1. $(\underline{1011}\ \underline{0100}\ \underline{0011})_2 = (B43)_{16}$

2. $(\underline{10}\ \underline{1010}\ \underline{0001})_2 = (2A1)_{16}$

3. $(.\underline{1000}\ \underline{010})_2 = (.84)_{16}$

4. $(\underline{101}\ .\ \underline{0101}\ \underline{111})_2 = (5.5E)_{16}$

Hexadecimal-to-Binary Conversion

- Translate every hexadecimal digit into its 4-bit binary equivalent.
  - Discard leading and trailing zeros if desired.
- Examples:

$(3A5)_{16} = (0011\ 1010\ 0101)_2$

$(12.3D)_{16} = (0001\ 0010\ .\ 0011\ 1101)_2$

$(1.8)_{16} = (0001\ .\ 1000)_2$

Number Representation

Unsigned and Signed numbers

Unsigned Binary Numbers

- An n-bit binary number

  $B = b_{n-1}b_{n-2} \dots. B_2b_1b_0$

  (2^n distinct combinations are possible, 0 to 2^n–1)

- For n = 3, there are 8 distinct combinations.
  - 000, 001, 010, 011, 100, 101, 110, 111

- Range of numbers that can be represented

  n=8 → 0 to 2^8–1 (255)

  n=16 → 0 to 2^{16}–1 (65535)

  n=32 → 0 to 2^{32}–1 (4294967295)

Signed Integer Representation

- ***Many of the numerical data items that are used in a program are signed (positive or negative).***
  - ***Question:: How to represent sign?***
- ***Three possible approaches:***

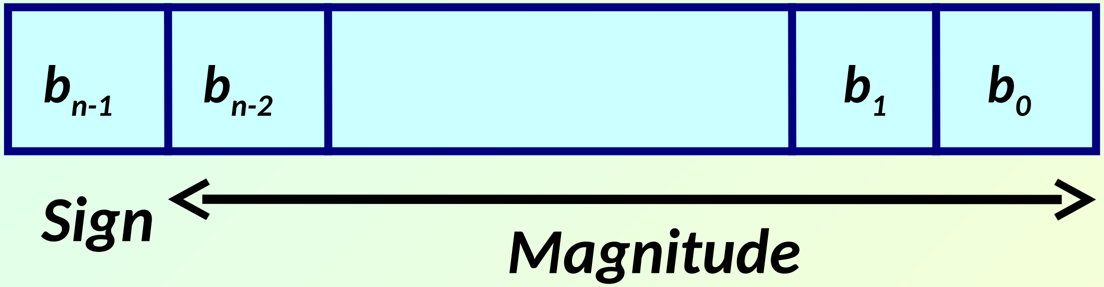
  a) ***Sign-magnitude representation***

  b) ***One's complement representation***

  c) ***Two's complement representation***

Sign-magnitude Representation

- *For an n-bit number representation*
  - *The most significant bit (MSB) indicates sign*
    - *0 → positive*
    - *1 → negative*
  - *The remaining n-1 bits represent magnitude.*



Example :: n=4

0000 → +0	1000 → -0
0001 → +1	1001 → -1
0010 → +2	1010 → -2
0011 → +3	1011 → -3
0100 → +4	1100 → -4
0101 → +5	1101 → -5
0110 → +6	1110 → -6
0111 → +7	1111 → -7

15 distinct numbers can be represented

Contd.

- **Range of numbers that can be represented:**

  Maximum :: $+(2^{n-1} - 1)$

  Minimum :: $-(2^{n-1} - 1)$

- **A problem:**

  Two different representations of zero.

  $+0 \rightarrow 0\ 000\ldots.0$

  $-0 \rightarrow 1\ 000\ldots.0$

One's Complement Representation

- **Basic idea:**
  - Positive numbers are represented exactly as in sign-magnitude form.
  - Negative numbers are represented in 1's complement form.
- **How to compute the 1's complement of a number?**
  - Complement every bit of the number (1→0 and 0→1).
  - MSB will indicate the sign of the number.

    0 → positive

    1 → negative

Example :: n=4

0000 → +0	*1000 → -7*
0001 → +1	*1001 → -6*
0010 → +2	*1010 → -5*
0011 → +3	*1011 → -4*
0100 → +4	*1100 → -3*
0101 → +5	*1101 → -2*
0110 → +6	*1110 → -1*
0111 → +7	*1111 → -0*

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 1's complement of 0100 = 1011

Contd.

- ***Range of numbers that can be represented:***

  Maximum :: + $(2^{n-1} – 1)$

  Minimum :: – $(2^{n-1} – 1)$

- ***A problem:***

  Two different representations of zero.

  +0 → 0 000....0

  -0 → 1 111....1

- ***Advantage of 1's complement representation***
  - ***Subtraction can be done using addition.***
  - ***Leads to substantial saving in circuitry.***
  - ***Sign extension is possible to increase number of bits to represent.***

    -3 = 1100 (in 4 bits) = 1111 1100 (in 8 bits)

    +3 = 0011 (in 4 bits) = 0000 0011 (in 8 bits)

Two's Complement Representation

- **Basic idea:**
  - Positive numbers are represented exactly as in sign-magnitude form.
  - Negative numbers are represented in 2's complement form.
- **How to compute the 2's complement of a number?**
  - Complement every bit of the number (1→0 and 0→1), and then **add 1** to the resulting number.
  - MSB will indicate the sign of the number.

    0 → positive

    1 → negative

Example :: n=4

0000 → +0	1000 → -8
0001 → +1	1001 → -7
0010 → +2	1010 → -6
0011 → +3	1011 → -5
0100 → +4	1100 → -4
0101 → +5	1101 → -3
0110 → +6	1110 → -2
0111 → +7	1111 → -1

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 2's complement of 0100 = 1011 + 1 = 1100

Contd.

- **Range of numbers that can be represented:**

  Maximum :: $+ (2^{n-1} - 1)$

  Minimum :: $- 2^{n-1}$

- **Advantage:**
  - Unique representation of zero.
  - Subtraction can be done using addition.
  - Leads to substantial saving in circuitry.
  - Sign extension is possible to increase number of bits to represent.

    -3 = 1101 (in 4 bits) = 1111 1101 (in 8 bits)

    +3 = 0011 (in 4 bits) = 0000 0011 (in 8 bits)

- **Almost all computers today use the 2's complement representation for storing negative numbers.**

Contd.

- *In C (typical values):*
  - *short int*
    - *16 bits* ➔ $+ (2^{15}-1)$ to -2^{15}
  - *int*
    - *32 bits* ➔ $+ (2^{31}-1)$ to -2^{31}
  - *long int*
    - *64 bits* ➔ $+ (2^{63}-1)$ to -2^{63}

Binary operations

Addition
Subtraction using addition

Binary addition

- **Rules for adding two bits:**

  0 + 0 = 0

  0 + 1 = 1

  1 + 0 = 1

  1 + 1 = 10, i.e., 0 with carry of 1

- **Addition examples for unsigned numbers:**

```
0 1 0 1      0 1 1 1        1 0 0 1
0 0 0 1      0 0 1 1        1 0 1 0
=======      =======      =========
0 1 1 0      1 0 1 0      1 0 0 1 1
```

Carry

Subtraction Using Addition :: 1's Complement

- How to compute A – B ?
  - Compute the 1's complement of B (say, B_1).
  - Compute $R = A + B_1$
  - If the carry obtained after addition is '1'
    - Add the carry back to R (called end-around carry).
    - That is, R = R + 1.
    - The result is a positive number.

    Else
    - The result is negative, and is in 1's complement form.

Example 1 :: 6 – 2

1's complement of 2 = 1101

```
       6 :: 0 1 1 0    A
      −2 :: 1 1 0 1    B1
             -------
           1 0 0 1 1   R
                   1
           ---------
             0 1 0 0  → +4
```

End-around carry

Assume 4-bit representations.

Since there is a carry, it is added back to the result.

The result is positive.

Example 2 :: 3 – 5

1's complement of 5 = 1010

3	::	0	0	1	1	A
−5	::	1	0	1	0	B_1
		1	1	0	1	R

⇩

−2

Assume 4-bit representations.

Since there is no carry, the result is negative.

1101 is the 1's complement of 0010, that is, it represents −2.

Subtraction Using Addition :: 2's Complement

- ***How to compute A – B ?***
  - ***Compute the 2's complement of B (say, B_2).***
  - ***Compute $R = A + B_2$***
  - ***If the carry obtained after addition is '1'***
    - ***Ignore the carry.***
    - ***The result is a positive number.***

    Else

    - ***The result is negative, and is in 2's complement form.***

Example 1 :: 6 – 2

2's complement of 2 = 1101 + 1 = 1110

6	::		0	1	1	0	A
−2	::		1	1	1	0	B_2
		1	0	1	0	0	R

Ignore carry

+4

Assume 4-bit representations.

Presence of carry indicates that the result is positive.

No need to add the end-around carry like in 1's complement.

Example 2 :: 3 – 5

2's complement of 5 = 1010 + 1 = 1011

3	::	0	0	1	1	A
−5	::	1	0	1	1	B_2
		1	1	1	0	R

−2

Assume 4-bit representations.

Since there is no carry, the result is negative.

1110 is the 2's complement of 0010, that is, it represents –2.

Floating-point number representation

Floating-point Numbers

- The representations discussed so far applies only to integers.
  - Cannot represent numbers with fractional parts.
- We can assume a decimal point before a 2's complement number.
  - In that case, pure fractions (without integer parts) can be represented.
- We can also assume the decimal point somewhere in between.
  - This lacks flexibility.
  - Very large and very small numbers cannot be represented.

Representation of Floating-Point Numbers

- *A floating-point number F is represented by a doublet <M,E> :*

  $F = M \times B^E$

  - *B : exponent base (usually 2)*
  - *M : mantissa*
  - *E : exponent*

  – *M is usually represented in 2's complement form, with an implied decimal point before it.*

- *For example,*

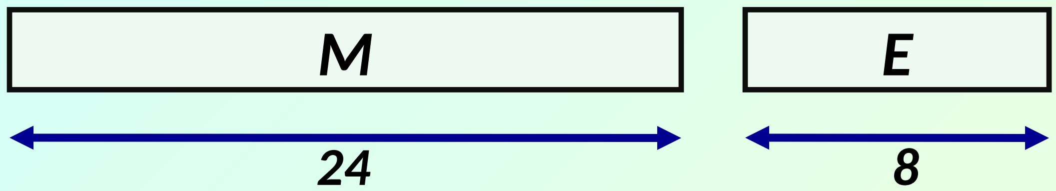
  In decimal,

  0.235×10^6

  In binary,

  0.101011×2^{0110}

Example :: 32-bit representation



- M represents a 2's complement fraction

  $1 > M > -1$

- E represents the exponent (in 2's complement form)

  $127 > E > -128$

- **Points to note:**
  - The number of significant digits depends on the number of bits in M.
    - 6 significant digits for 24-bit mantissa.
  - The range of the number depends on the number of bits in E.
    - 10^{38} to 10^{-38} for 8-bit exponent.

A Warning

- The representation for floating-point numbers as shown is just for illustration.
- The actual representation is a little more complex.
- In C:
  - float :: 32-bit representation
  - double :: 64-bit representation

Representation of Characters and Strings

Representation of Characters

- Many applications have to deal with non-numerical data
  - Characters and strings.
  - There must be a standard mechanism to represent alphanumeric and other characters in memory.
- Three standards in use:
  - Extended Binary Coded Decimal Interchange Code (EBCDIC)
    - Used in older IBM machines.
  - American Standard Code for Information Interchange (ASCII)
    - Most widely used today.
  - UNICODE
    - Used to represent all international characters.
    - Used by Java.

ASCII Code

- Each individual character is numerically encoded into a unique 7-bit binary code.
  - A total of 2^7 or 128 different characters.
  - A character is normally encoded in a byte (8 bits), with the MSB not been used.
- The binary encoding of the characters follow a regular ordering.
  - Digits are ordered consecutively in their proper numerical sequence (0 to 9).
  - Letters (uppercase and lowercase) are arranged consecutively in their proper alphabetic order.

Some Common ASCII Codes

‘A’ :: 41 (H) 65 (D)

‘B’ :: 42 (H) 66 (D)

....

‘Z’ :: 5A (H) 90 (D)

‘a’ :: 61 (H) 97 (D)

‘b’ :: 62 (H) 98 (D)

....

‘z’ :: 7A (H) 122 (D)

‘0’ :: 30 (H) 48 (D)

‘1’ :: 31 (H) 49 (D)

....

‘9’ :: 39 (H) 57 (D)

‘(‘ :: 28 (H) 40 (D)

‘+’ :: 2B (H) 43 (D)

‘?’ :: 3F (H) 63 (D)

‘\n’ :: 0A (H) 10 (D)

‘\0’ :: 00 (H) 00 (D)

Character Strings

- Two ways of representing a sequence of characters in memory.
  - The first location contains the number of characters in the string, followed by the actual characters.

5	H	e	l	l	o

  - The characters follow one another, and is terminated by a special delimiter.

H	e	l	l	o	$\perp$

String Representation in C

- *In C, the second approach is used.*
  - *The ‘\0’ character is used as the string delimiter.*
- *Example:*

  “Hello” →

H	e	l	l	o	‘\0’

- *A null string “” occupies one byte in memory.*
  - *Only the ‘\0’ character.*