Assignment 1: Algorithmic Game Theory

Palash Dey
Indian Institute of Technology, Kharagpur

September 5, 2020

1. In a normal form game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ if a pure strategy $s_i \in S_i$ for some player i is strongly dominated by some mixed strategy $\sigma_i \in \Delta(S_i)$, then in every MSNE of the game, player i chooses the strategy s_i with probability 0.

2. Give an example of a normal form game which does not have any MSNE.

3. Compute an MSNE for the matching pennies and the rock-paper-scissor games. Prove that these games has unique MSNEs.

4. In a normal form game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a strategy $s_i \in S_i$ for player i is called *strongly dominated* if there exists a mixed strategy $\sigma_i \in \Delta(S_i)$ for the player i which strongly dominates s_i. That is,

$$u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i}) \forall s_{-i} \in S_{-i}$$

Prove that a strongly dominated strategy cannot have a non-zero probability in any MSNE of the game. We can use this result repeatedly to reduce the game. This strategy is called *iterative elimination of strongly dominated strategies*. Using iterative elimination of strongly dominated strategies, find all equilibrium of Game 1 (Source: [Mye97]).

Player 2

		A	B	C
Player 1	A	$(2, 3)$	$(3, 0)$	$(0, 1)$
	B	$(0, 0)$	$(1, 6)$	$(4, 2)$

Game 1

Player 2

		A	B	C
Player 1	A	$(1, 1)$	$(1, 1)$	$(0, 0)$
	B	$(0, 0)$	$(1, 2)$	$(1, 2)$

Game 2

5. In a normal form game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a strategy $s_i \in S_i$ for player i is called *weakly dominated* if there exists a mixed strategy $\sigma_i \in \Delta(S_i \setminus \{s_i\})$ for the player i which always less or equal utility to player i irrespective of what others play and there exists a strategy profile of other players where s_i gives strictly less utility to player i. That is,

$$u_i(s_i, s_{-i}) \leqslant u_i(\sigma_i, s_{-i}) \forall s_{-i} \in S_{-i}$$

Prove that, for every weakly dominated strategy s_i for player i, there exists an MSNE where player i never plays s_i. We can use this result repeatedly to reduce the game if our goal is to find one MSNE (since there can be an MSNE where player i plays a weakly dominated strategy). This strategy is called *iterative elimination of weakly dominated strategies*. Using iterative elimination of strongly dominated strategies, find all equilibrium of Game 2 (Source: [Osb04]).

6. *Guessing game:* There are 56 students in the Algorithmic Game Theory class in IIT KGP in 2019. They play the following game. They guess any natural number in the interval $[0, 100]$. The student whose

guess is closest to the 2/3 of the average of all the guesses wins the game and receives a cash prize of 1000 rupees; in case of ties, the prize money gets shared equally. Formulate this game in normal form. Find an MSNE of this game. Prove that the MSNE found is unique.

7. Compute all MSNEs of the following game (Source: [Nar14]).

   ▷ The set of players $(N) : \{1, 2\}$

   ▷ The set of strategies: $S_i = \{A, B\}$ for every $i \in [2]$

   ▷ Payoff matrix:

		Player 2	
		A	B
Player 1	A	$(20, 0)$	$(0, 10)$
	B	$(0, 90)$	$(20, 0)$

8. Compute all MSNEs of the following coordination game.

   ▷ The set of players $(N) : \{1, 2\}$

   ▷ The set of strategies: $S_i = \{A, B\}$ for every $i \in [2]$

   ▷ Payoff matrix:

		Player 2	
		A	B
Player 1	A	$(10, 10)$	$(0, 0)$
	B	$(0, 0)$	$(1, 1)$

9. Compute all MSNEs of the tragedy of commons game.

   ▷ The set of players $(N) : \{1, 2, \ldots, n\}$ (we denote this set by $[n]$)

   ▷ The set of strategies: $S_i = \{0, 1\}$ for every $i \in [n]$

   ▷ Utility:

   $$u_i(s_1, \ldots, s_i, \ldots, s_n) = s_i - \left[\frac{5(s_1 + \cdots + s_n)}{n}\right]$$

10. Prove that bidding valuations does not always form a PSNE in the first price auction. That is, give an example of a first price auction scenario where bidding valuation is not a PSNE.

11. Compute an MSNE, if any, for the following game (Source: [Nar14]).

    ▷ The set of players $(N) : \{1, 2\}$

    ▷ The set of strategies: $S_1 = [0, 1], S_2 = [3, 4]$

    ▷ Utility: $u_1(x, y) = -u_2(x, y) = |x - y|, \forall (x, y) \in [0, 1] \times [3, 4]$

12. Consider a road networks shown in Figure 1. The numbers on the edges indicate the time (in minutes say) one requires to traverse the edges. The variable x denote the number of commuters using that edge. Suppose there are 100 people who want to reach the vertex B from vertex A. Write the strategic form games corresponding to the networks in Figure 1 and find the PSNEs for both the games.

13. Let $\mathcal{A}$ be a $n \times n$ matrix of a matrix game. Prove that, if $\mathcal{A}$ is anti-symmetric, then the value of the row player in mixed strategies is 0.

14. Let $\mathcal{A}$ be a $m \times n$ matrix of a matrix game. Assume that (i, j) and (h, k) are two PSNEs of the matrix game. Then prove that (i, k) and (h, j) are also two PSNEs of the matrix game.

15. Let $\mathcal{A}$ be a $n \times n$ matrix of a matrix game. Assume $\mathcal{A}$ is a latin square; that is, each row and each column of $\mathcal{A}$ is a permutation of $\{1, 2, \ldots, n\}$. Compute a PSNE of the corresponding matrix game if it exists.

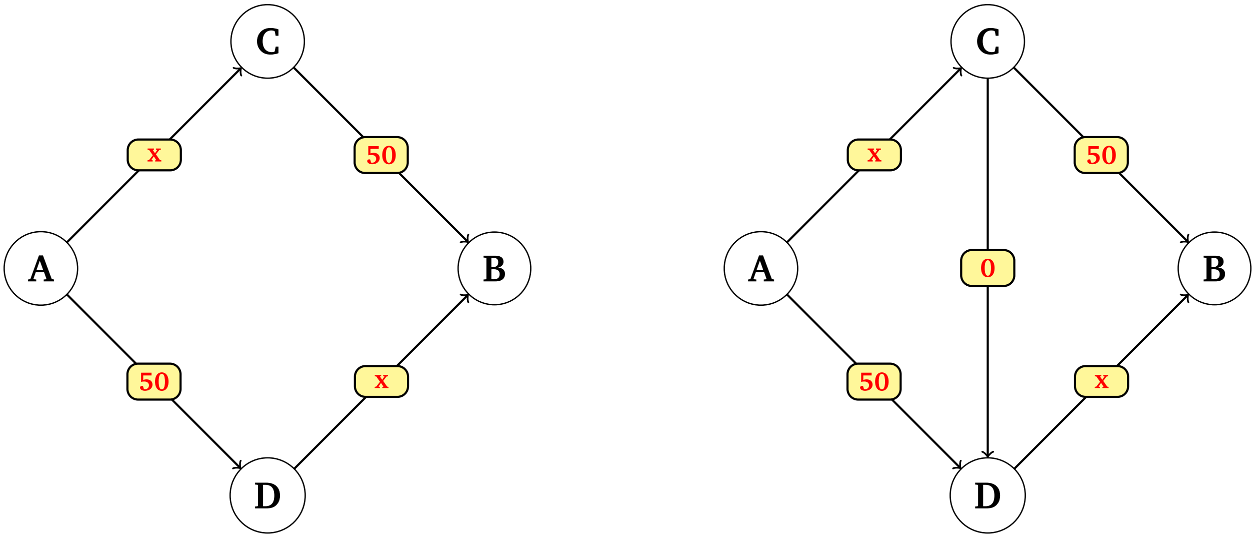

Figure 1: Braess paradox.

16. Suppose in a matrix game, the players have 3 strategies each. Which numbers among $\{0, 1, 2, \dots, 9\}$ cannot be the total number PSNEs in the matrix game.

17. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite sets, and $f : \mathcal{A} \times \mathcal{B} \longrightarrow \mathbb{R}$ be any arbitrary function. Then prove that,

$$\max_{a\in\mathcal{A}} \min_{b} \in \mathcal{B} f(a, b) \leqslant \min_{b\in\mathcal{B}} \max_{a\in\mathcal{A}} f(a, b)$$

18. Given an $m \times n$ matrix $\mathcal{A}$, an entry $a_{i,j}$ is called a *saddle point* of $\mathcal{A}$ if $a_{i,j}$ is simultaneously a maximum for the j-th column and minimum for the i-th row. Prove that $a_{i,j}$ is a saddle point of $\mathcal{A}$ if and only if (i, j) is a PSNE for the corresponding matrix game.

19. Let $\underline{v}$ and $\bar{v}$ be respectively the maxmin and minmax value of a matrix $\mathcal{A}$ in pure strategies. Prove that the corresponding matrix game has a PSNE if and only if $\mathcal{A}$ has a saddle point.

20. Let $\underline{v}$ and $\bar{v}$ be respectively the maxmin and minmax value of a matrix $\mathcal{A}$ in pure strategies. Prove that $\underline{v} = \bar{v}$ if and only if $\mathcal{A}$ has a saddle point.

21. Suppose player i has a pure strategy s_i that us chosen with positive probability in every maxmin strategy for that player. Prove that s_i is not weakly dominated by any other pure or mixed strategy.

References

[Mye97] Roger B. Myerson. *Game theory - Analysis of Conflict*. Harvard University Press, 1997.

[Nar14] Y. Narahari. *Game Theory and Mechanism Design*. World Scientific Publishing Company Pte. Limited, 2014.

[Osb04] Martin J. Osborne. *An introduction to game theory*. Oxford Univ. Press, New York, NY [u.a.], 2004.