Max-flow min-cut theorem using LP duality

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Let $G = (V, E)$ be a directed graph. Each edge $e \in E$ has a non-negative capacity c_e. s and t are two designated vertices in G. The task is to compute the max-flow from s to t. To this end, we add a special directed edge from t to s, with infinite capacity, which basically means that the capacity is so large that you can never violate the capacity constraint. In other words we will not impose any capacity constraint on this edge. In the resultant graph we will enforce flow conservation constraints on <u>all</u> vertices, including s and t, and will maximize the flow along the edge (t, s). It is easy to see that the value of a maximum flow along (t, s) in this graph is equal to the value of a maximum s-t flow in the original graph. Check that the value of the following linear program L (we denote it by $\mathsf{val}(L)$) is equal to the max-flow of G.

maximize $f_{t,s}$

subject to:

for each edge $e \neq (t, s), f_e \leqslant c_e,$ (capacity constraints)

for each vertex v $\sum_{(u,v) \in E} f_{u,v} - \sum_{(v,w) \in E} f_{v,w} \leqslant 0.$ (flow-conservation constraints)

for each edge $e, f_e \geqslant 0$ (non-negativity constraints)

Notice that we actually need equality (instead of $\leqslant$) in the flow-conservation constraints. However, if we enforce $\leqslant$ for all the constraints, it follows that equality is actually enforced for all the constraints. We did it in class: add the left hand sides over various vertices v and check for yourself that the sum becomes 0. The only way in which the sum of some non-positive numbers can be 0 is that each number in the sum is equal to 0.

Next, let us write the dual L' of L. Let γ_e and μ_v be the dual variables that correspond to the capacity and flow-conservation constraints in L respectively.

minimize $\sum_{e \in E} c_e \gamma_e$

subject to:

for each edge $e = (u, v) \neq (t, s), \gamma_e - \mu_u + \mu_v \geqslant 0,$

$\mu_s - \mu_t \geqslant 1.$

for each edge $e \neq (t, s)$ and vertex $v, \gamma_e, \mu_v \geqslant 0$ (non-negativity constraints)

We will use the following fact.

Fact 1. *The coefficient matrix of* L' *is totally unimodular.*

Fact 1 is true because

1. The coefficient matrix of L is totally unimodular. The proof is similar to that of the Bipartite matching LP that we did in class. Please do it yourself.
2. The coefficient matrix of the dual is the transpose of the coefficient matrix of the primal (check yourself.).

Claim 1. $val(L')$ *is equal to the capacity of a minimum* s-t *cut in* G.

Note that Claim 1 implies the max-flow min-cut theorem because

$$\begin{aligned}\text{max-flow} &= \mathsf{val}(L)\\ &= \mathsf{val}(L') \quad \text{(By strong duality)}\\ &= \text{capacity of a minimum cut (By Claim 1)}.\end{aligned}$$

Proof of Claim 1. We split the proof into two parts.

(i) $\mathsf{val}(L') \leqslant$ the capacity of a minimum s-t cut in G.

Let $(A, V \setminus S)$ is a minimum s-t cut in G, and $s \in A, t \in V \setminus A$. From this cut, let's set the dual variables as follows:

$$\gamma_e = \begin{cases} 1 & \text{if } e = (u,v), u \in A, v \in V \setminus A,\\ 0 & \text{otherwise.}\end{cases}$$

$$\mu_v = \begin{cases} 1 & \text{if } v \in A\\ 0 & \text{otherwise.}\end{cases}$$

□

These values satisfy all the dual constraints (substitute and check for yourself). Thus we have,

$$\begin{aligned}\mathsf{val}(L') &\leqslant \sum_{e \in E} c_e \gamma_e\\ &= \sum_{e=(u,v): u \in A, v \in V \setminus A} c_e\\ &= \text{capacity of } (A, V \setminus A).\\ &= \text{capacity of a minimum } s\text{-}t \text{ sut in } G.\end{aligned}$$

(ii) The capacity of a minimum s-t cut in $G \leqslant \mathsf{val}(L')$

Let $(\gamma_e^*, \mu_v^* : e \in E \setminus \{(t,s)\}, v \in V)$ be an integral optimal solution for L'[1]. Define $E' := \{e \in E : \gamma_e^* > 0.\}$ Consider the graph $G' := (V, E \setminus E')$ obtained by removing the edges in E' from G.

[1]Recall rom Fact 1 that the coefficient matrix of L' is totally unimodular; so L' has an integral optimum.

Claim 2. *There is no* s-t *path in* G'.

Let us first assume Claim 2 and finish the proof, and then prove Claim 2. Claim 2 suggests the following cut. Define $A := \{\text{All vertices reachable from } s \text{ in } G'\}$. Note that by Claim 2, $t \notin A$, and hence $(A, V \setminus A)$ is an s-t cut. Now, let $e = (u, v) \in E$ be such that $\gamma_e^* = 0$. We claim that e cannot be a "cut-edge", i.e., it is not true that $u \in A$ and $v \in V \setminus A$. To see this, notice that $e \notin E'$ and hence e is present in G'. If $u \in A$, then by the definition of A, u is reachanbel from s in G'. Now, since $e = (u, v)$ is also present in G', it follows that v is also reachable from s in G', and hence $v \in A$. We thus have that for every cut-edge e, $\gamma_e^* > 0$. Since γ_e^* is integral, we have that $\gamma_e^* \geqslant 1$. We thus have

$$\begin{aligned}
&\text{Capacity of a minimum } s\text{-}t \text{ cut in } G \\
&\leqslant \text{capacity of } (A, V \setminus A) \\
&= \sum_{(u,v) \in E : u \in A, v \in V \setminus A} c_e \\
&\leqslant \sum_{(u,v) \in E : u \in A, v \in V \setminus A} \gamma_e^* c_e \quad (\text{Since } \gamma_e^* \geqslant 1 \text{ for every cut-edge } e) \\
&= \mathsf{val}(L').
\end{aligned}$$

We are thus left with the task of proving Claim 2.

Proof of Claim 2. Towards a contradiction, assume that there is a path $(s, w_1, \ldots, w_\ell, t)$ in G'. We have that,

$$\begin{aligned}
1 &\leqslant \mu_s^* - \mu_t^* \qquad (\text{from the last dual constraint}) \\
&= (\mu_s^* - \mu_{w_1}^*) + (\mu_{w_1}^* - \mu_{w_2}^*) + \ldots + (\mu_{w_{\ell-1}}^* - \mu_{w_\ell}^*) + (\mu_{w_\ell}^* - \mu_t^*) \\
&\leqslant \gamma_{s,w_1} + \gamma_{w_1,w_2} + \ldots + \gamma_{w_{\ell-1},w_\ell} + \gamma_{w_\ell,t} \\
&\qquad\qquad (\text{By the dual constraints for the respective edges})
\end{aligned}$$

Now since $(s, w_1), (w_1, w_2), \ldots, (w_{\ell-1}, w_\ell), (w_\ell, t)$ are edges in $E \setminus E'$, the corresponding γ^* values are all non-positive. From the non-negativity constraints of L' we know that the γ^* values are also non-negative. We thus conclude that all these γ^* values are 0. Hence their sum is also 0. We thus have $1 \leqslant 0$ which is a contradiction. This proves the Claim. $\square$