# CS60007 Algorithm Design and Analysis 2018
# Supplementary for Lecture 1

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July 19, 2018

**Lemma 1.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected weighted graph, $\mathcal{S} \subset \mathcal{V}$ a subset of vertices with $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \neq \mathcal{V}$, $e \in \mathcal{E}$ an edge of minimum weight in the cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$. Then there exists a minimum spanning tree of $\mathcal{G}$ which includes the edge $e$.*

*Proof.* Let $\mathcal{T}$ be a minimum spanning tree of $\mathcal{G}$ and $e = \{u, v\}$. If $\mathcal{T}$ already includes $e$, then we have nothing to prove. So, let us assume that $\mathcal{T}$ does not include the edge $e$. Let us consider the subgraph $\mathcal{F} = \mathcal{T} \cup \{e\}$ of $\mathcal{G}$. By the definition of trees, there exists a cycle $\mathcal{C}$ in $\mathcal{F}$ which includes the edge $e$. Now, since $\mathcal{C} \cap \mathcal{S} \neq \emptyset$ and $\mathcal{C} \cap (\mathcal{V} \setminus \mathcal{S}) \neq \emptyset$, there exists an edge $f$ in $\mathcal{C}$ other than $e$ that belongs to the cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$. By the definition of $e$, we have weight of $f$ is at least the weight of $e$. Also, $\mathcal{T}' = \mathcal{F} \setminus \{f\}$ is a spanning tree of $\mathcal{G}$ since $\mathcal{T}$ is a spanning tree of $\mathcal{G}$. We observe that the weight of $\mathcal{T}'$ is at most the weight of $\mathcal{T}$ since the weight of $e$ is at most the weight of $f$. However, since $\mathcal{T}$ is a minimum spanning tree of $\mathcal{G}$, $\mathcal{T}'$ is also a minimum spanning tree of $\mathcal{G}$. This proves the result since $\mathcal{T}'$ includes $e$. □