Practice Problem Set III

Probabilistic Graphical Models

(From the text books by Kevin Murphy, and David Barber)

Exercise 10.3 Markov blanket for a DGM

Prove that the full conditional for node i in a DGM is given by

$$p(X_i|X_{-i}) \propto p(X_i|Pa(X_i)) \prod_{Y_j \in ch(X_i)} p(Y_j|Pa(Y_j))$$

where $ch(X_i)$ are the children of X_i and $Pa(Y_j)$ are the parents of Y_j.

Exercise 10.5 Bayes nets for a rainy day

(Source: Nando de Freitas.). In this question you must model a problem with 4 binary variables: G ="gray", V ="Vancouver", R ="rain" and S ="sad". Consider the directed graphical model describing the relationship between these variables shown in Figure 10.15(a).

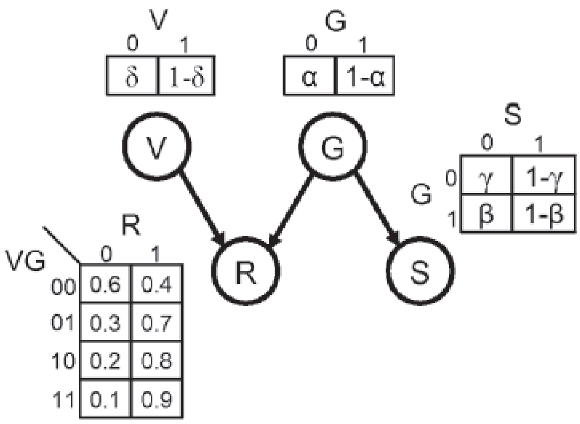
a. Write down an expression for $P(S=1|V=1)$ in terms of $\alpha, \beta, \gamma, \delta$.

b. Write down an expression for $P(S=1|V=0)$. Is this the same or different to $P(S=1|V=1)$? Explain why.

c. Find maximum likelihood estimates of α, β, γ using the following data set, where each row is a training case. (You may state your answers without proof.)

V	G	R	S
1	1	1	1
1	1	0	1
1	0	0	0

(10.61)



(a)

Figure 10.15(a)

Exercise 10.6 Fishing nets

(Source: (Duda et al. 2001)..) Consider the Bayes net shown in Figure 10.15(b). Here, the nodes represent the following variables

$$X_1 \in \{\text{winter, spring, summer, autumn}\},\ X_2 \in \{\text{salmon, sea bass}\} \quad (10.62)$$

$$X_3 \in \{\text{light, medium, dark}\},\ X_4 \in \{\text{wide, thin}\} \quad (10.63)$$

The corresponding conditional probability tables are

$$p(x_1) = \begin{pmatrix} .25 & .25 & .25 & .25 \end{pmatrix},\ p(x_2|x_1) = \begin{pmatrix} .9 & .1 \\ .3 & .7 \\ .4 & .6 \\ .8 & .2 \end{pmatrix} \quad (10.64)$$

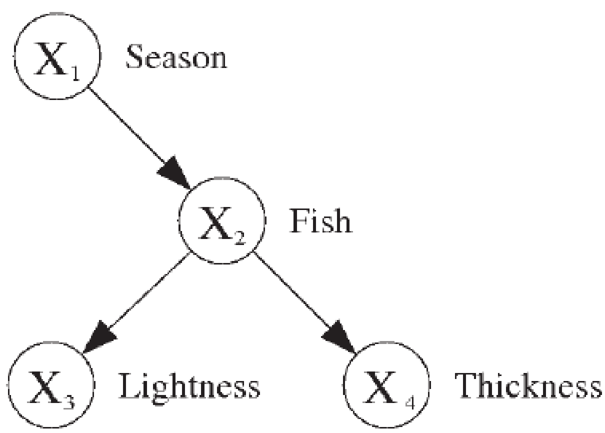
$$p(x_3|x_2) = \begin{pmatrix} .33 & .33 & .34 \\ .8 & .1 & .1 \end{pmatrix},\ p(x_4|x_2) = \begin{pmatrix} .4 & .6 \\ .95 & .05 \end{pmatrix} \quad (10.65)$$

Note that in $p(x_4|x_2)$, the rows represent x_2 and the columns x_4 (so each row sums to one and represents the child of the CPD). Thus $p(x_4 = \text{thin}|x_2 = \text{sea bass}) = 0.05$, $p(x_4 = \text{thin}|x_2 = \text{salmon}) = 0.6$, etc.

Answer the following queries. You may use matlab or do it by hand. In either case, show your work.

a. Suppose the fish was caught on December 20 — the end of autumn and the beginning of winter — and thus let $p(x_1) = (.5,\ 0,\ 0,\ .5)$ instead of the above prior. (This is called **soft evidence**, since we do not know the exact value of X_1, but we have a distribution over it.) Suppose the lightness has not been measured but it is known that the fish is thin. Classify the fish as salmon or sea bass.

b. Suppose all we know is that the fish is thin and medium lightness. What season is it now, most likely? Use $p(x_1) = \begin{pmatrix} .25 & .25 & .25 & .25 \end{pmatrix}$



(b)

Figure 10.15(b)

Exercise 19.5 Full conditional in an Ising model

Consider an Ising model

$$p(x_1, \ldots, x_n|\boldsymbol{\theta}) = \frac{1}{Z(\boldsymbol{\theta})} \prod_{<ij>} \exp(J_{ij} x_i x_j) \prod_{i=1}^{n} \exp(h_i x_i) \tag{19.125}$$

where $< ij >$ denotes all unique pairs (i.e., all edges), $J_{ij} \in \mathbb{R}$ is the coupling strength (weight) on edge $i - j$, $h_i \in \mathbb{R}$ is the local evidence (bias term), and $\boldsymbol{\theta} = (\mathbf{J}, \mathbf{h})$ are all the parameters.

If $x_i \in \{0, 1\}$, derive an expression for the full conditional

$$p(x_i = 1|\mathbf{x}_{-i}, \boldsymbol{\theta}) = p(x_i = 1|\mathbf{x}_{nb_i}, \boldsymbol{\theta}) \tag{19.126}$$

where $\mathbf{x}_{-i}$ are all nodes except i, and nb_i are the neighbors of i in the graph. Hint: you answer should use the sigmoid/ logistic function $\sigma(z) = 1/(1 + e^{-z})$. Now suppose $x_i \in \{-1, +1\}$. Derive a related expression for $p(x_i|\mathbf{x}_{-i}, \boldsymbol{\theta})$ in this case. (This result can be used when applying Gibbs sampling to the model.)

Exercise 20.3 Message passing on a tree

Consider the DGM in Figure 20.10 which represents the following fictitious biological model. Each G_i represents the genotype of a person: $G_i = 1$ if they have a healthy gene and $G_i = 2$ if they have an unhealthy gene. G_2 and G_3 may inherit the unhealthy gene from their parent G_1. $X_i \in \mathbb{R}$ is a continuous measure of blood pressure, which is low if you are healthy and high if you are unhealthy. We define the CPDs as follows

$$p(G_1) = [0.5, 0.5] \tag{20.73}$$

$$p(G_2|G_1) = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix} \tag{20.74}$$

$$p(G_3|G_1) = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix} \tag{20.75}$$

$$p(X_i|G_i = 1) = \mathcal{N}(X_i|\mu = 50, \sigma^2 = 10) \tag{20.76}$$

$$p(X_i|G_i = 2) = \mathcal{N}(X_i|\mu = 60, \sigma^2 = 10) \tag{20.77}$$

The meaning of the matrix for $p(G_2|G_1)$ is that $p(G_2 = 1|G_1 = 1) = 0.9$, $p(G_2 = 1|G_1 = 2) = 0.1$, etc.

a. Suppose you observe $X_2 = 50$, and X_1 is unobserved. What is the posterior belief on G_1, i.e., $p(G_1|X_2 = 50)$?

b. Now suppose you observe $X_2 = 50$ amd $X_3 = 50$. What is $p(G_1|X_2, X_3)$? Explain your answer intuitively.

c. Now suppose $X_2 = 60$, $X_3 = 60$. What is $p(G_1|X_2, X_3)$? Explain your answer intuitively.

d. Now suppose $X_2 = 50$, $X_3 = 60$. What is $p(G_1|X_1, X_2)$? Explain your answer intuitively.

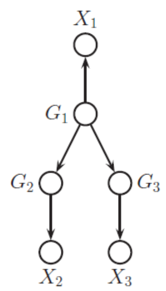


Figure 20.10 A simple DAG representing inherited diseases.

Exercise 23 ([128]). *Consider the network in fig(3.16), which concerns the probability of a car starting.*

$$
\begin{array}{ll}
p(b = \textsf{bad}) = 0.02 & p(f = \textsf{empty}) = 0.05 \\
p(g = \textsf{empty}|b = \textsf{good}, f = \textsf{not empty}) = 0.04 & p(g = \textsf{empty}|b = \textsf{good}, f = \textsf{empty}) = 0.97 \\
p(g = \textsf{empty}|b = \textsf{bad}, f = \textsf{not empty}) = 0.1 & p(g = \textsf{empty}|b = \textsf{bad}, f = \textsf{empty}) = 0.99 \\
p(t = \textsf{fa}|b = \textsf{good}) = 0.03 & p(t = \textsf{fa}|b = \textsf{bad}) = 0.98 \\
p(s = \textsf{fa}|t = \textsf{tr}, f = \textsf{not empty}) = 0.01 & p(s = \textsf{fa}|t = \textsf{tr}, f = \textsf{empty}) = 0.92 \\
p(s = \textsf{fa}|t = \textsf{fa}, f = \textsf{not empty}) = 1.0 & p(s = \textsf{fa}|t = \textsf{fa}, f = \textsf{empty}) = 0.99
\end{array}
$$

Calculate $P(f = \textsf{empty}|s = \textsf{no})$, the probability of the fuel tank being empty conditioned on the observation that the car does not start.

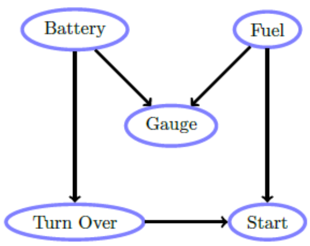


Figure 3.16: Belief Network of car not starting

Figure 3.16

Exercise 35. *A Belief Network models the relation between the variables oil, inf, eh, bp, rt which stand for the price of oil, inflation rate, economy health, British Petroleum Stock price, retailer stock price. Each variable takes the states* low, high, *except for bp which has states* low, high, normal. *The Belief Network model for these variables has tables*

$p(eh{=}\textsf{low}){=}0.2$	
$p(bp{=}\textsf{low}\|oil{=}\textsf{low}){=}0.9$	$p(bp{=}\textsf{normal}\|oil{=}\textsf{low}){=}0.1$
$p(bp{=}\textsf{low}\|oil{=}\textsf{high}){=}0.1$	$p(bp{=}\textsf{normal}\|oil{=}\textsf{high}){=}0.4$
$p(oil{=}\textsf{low}\|eh{=}\textsf{low}){=}0.9$	$p(oil{=}\textsf{low}\|eh{=}\textsf{high}){=}0.05$
$p(rt{=}\textsf{low}\|inf{=}\textsf{low},eh{=}\textsf{low}){=}0.9$	$p(rt{=}\textsf{low}\|inf{=}\textsf{low},eh{=}\textsf{high}){=}0.1$
$p(rt{=}\textsf{low}\|inf{=}\textsf{high},eh{=}\textsf{low}){=}0.1$	$p(rt{=}\textsf{low}\|inf{=}\textsf{high},eh{=}\textsf{high}){=}0.01$
$p(inf{=}\textsf{low}\|oil{=}\textsf{low},eh{=}\textsf{low}){=}0.9$	$p(inf{=}\textsf{low}\|oil{=}\textsf{low},eh{=}\textsf{high}){=}0.1$
$p(inf{=}\textsf{low}\|oil{=}\textsf{high},eh{=}\textsf{low}){=}0.1$	$p(inf{=}\textsf{low}\|oil{=}\textsf{high},eh{=}\textsf{high}){=}0.01$

1. *Draw a Belief Network for this distribution.*

2. *Given that BP stock price is normal and the retailer stock price is high, what is the probability that inflation is high?*

Exercise 43. *The undirected graph*  *represents a Markov Network with nodes x_1, x_2, x_3, x_4, x_5, counting clockwise around the pentagon with potentials $\phi(x_i, x_{1+mod(i,5)})$. Show that the joint distribution can be written as*

$$p(x_1, x_2, x_3, x_4, x_5) = \frac{p(x_1, x_2, x_5)p(x_2, x_4, x_5)p(x_2, x_3, x_4)}{p(x_2, x_5)p(x_2, x_4)} \tag{4.8.10}$$

and express the marginal probability tables explicitly as functions of the potentials $\phi(x_i, x_j)$.

Exercise 59. *Consider the following distribution:*

$$p(x_1, x_2, x_3, x_4) = \phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4) \tag{6.11.1}$$

1. *Draw a clique graph that represents this distribution and indicate the separators on the graph.*

2. *Write down an alternative formula for the distribution $p(x_1, x_2, x_3, x_4)$ in terms of the marginal probabilities $p(x_1, x_2)$, $p(x_2, x_3)$, $p(x_3, x_4)$, $p(x_2)$, $p(x_3)$*

Exercise 61. *Consider the distribution*

$$p(a, b, c, d, e, f, g, h, i) = p(a)p(b|a)p(c|a)p(d|a)p(e|b)p(f|c)p(g|d)p(h|e, f)p(i|f, g) \tag{6.11.3}$$

1. *Draw the Belief Network for this distribution.*

2. *Draw the moralised graph.*

3. *Draw the triangulated graph. Your triangulated graph should contain cliques of the smallest size possible.*

Exercise 62. *This question concerns the distribution*

$$p(a, b, c, d, e, f) = p(a)p(b|a)p(c|b)p(d|c)p(e|d)p(f|a, e) \tag{6.11.4}$$

1. *Draw the Belief Network for this distribution.*

2. *Draw the moralised graph.*

3. *Draw the triangulated graph. Your triangulated graph should contain cliques of the smallest size possible.*

4. *Draw a junction tree for the above graph*