Advanced Machine Learning
Class Test I Solutions

1. Let vector V_t denote the values of a set of stocks on the t-th day. The change of stock value is governed by the following model -

$$V_t = MV_{t-1} + \eta_t \quad \text{for} \quad t > 1,$$

where M is a given matrix and η_t is a zero mean Gaussian noise vector with covariance $\sigma^2 I$. Also, $p(V_1) = \mathcal{N}(0, \Sigma)$ is a Gaussian. Further, let Y_t be an economic index with linear relation -

$$Y_t = NV_t + \epsilon_t,$$

where N is known and ϵ_t is a zero mean Gaussian noise with covariance $\tau^2 I$. The τ and ϵ vectors are uncorrelated.

i. Show that $V_1, V2, \ldots, V_t$ is Gaussian distributed.

Ans: Given V_1 is a Gaussian. Then, $V_2 = MV_1 + \eta_2$ is also a Gaussian since sum of two Gaussian variable is a Gaussian. Hence, $p(V_1, V_2) = p(V_2|V1)p(V_1)$ is also a Gaussian (check by writing the algebraic form of the distribution). By repeating, $V_1, V_2, \ldots, V_t$ is jointly Gaussian.

ii. Show that the covariance matrix of $V_1, \ldots, V_t$ has elements of the form:

$$\text{cov}(V_t, V_t') = M^{t'-t}\Sigma \quad \text{if} \quad t \neq t', \quad \text{and} \quad M^t\Sigma(M^t)^T \quad \text{if} \quad t = t'.$$

Ans: By repeating,

$$V_t = M(MV_{t-2} + \eta_{t-1}) + \eta_t = M^2 V_{t-2} + M\eta_{t-1} + \eta_t = M^{t-1} + \sum_{i=2}^{t} M^{t-i}\eta_i$$

Since, $\text{Cov}(V\eta_i) = 0$ (independent noise) and $\text{Cov}(\eta_i\eta_j) = \sigma^2 I\delta_{i,j}$ (given), we have -

$$\text{Cov}(V_t', V_t) = \text{Cov}([M^{t'-1}V_1 + \sum_{i=2}^{t'} M^{t'-i}\eta_i] \ \ [M^{t-1}V_1 + \sum_{j=2}^{t} M^{t-j}\eta_j])$$

$$= M^{t'-1}\Sigma(M^{t-1})^T + \sigma^2 \sum_{i=2}^{t'}\sum_{j=2}^{t} \delta_{i,j} M^{t'-i}(M^{t-j})^T$$

$$== M^{t'-1}\Sigma(M^{t-1})^T + \sigma^2 \sum_{i=2}^{\min(t,t')} M^{t'-i}(M^{t-i})^T$$

The expression for covariance follows from the above equation.

iii. Explain if $V_1, V_2, \ldots, V_t$ is a Gaussian process.

Ans: Since joint distribution of $V_1, V_2, \ldots, V_t$ has been shown to be a Gaussian, any subset of these variables (marginal) are also Gaussian. Hence, by definition it is a Gaussian process.

iv. Show that $Y_1, Y_2, \ldots, Y_t$ is a Gaussian process.

Ans: Since V is jointly Gaussian and $Y_1, Y_2, \ldots, Y_t$ is linearly dependent on $V_1, V_2, \ldots, V_t$, then $Y_1, Y_2, \ldots, Y_t$ is also jointly Gaussian. (See transformation rules of Gaussian distributions.) Thus, $Y_1, Y_2, \ldots, Y_t$ is a Gaussian process.