Advanced graph theory: Homework 2: CS60047 Autumn 2023

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1. State Tutte's theorem precisely and illustrate multiple or unique "bad" sets for connected, even, undirected graphs that do not have a perfect matching. One example is $K_{1,3}$. Another example was worked out in class with 20 vertices and a maximum matching of only 9 edges.

2. Let $G'(V, E) \neq K_n$ be an n-vertex simple connected undirected graph where adding any additional edge e to G' would introduce a perfect matching in $G' + e$, given that G' has no perfect matching. Naturally G' has an even number of vertices.

   Apply Tutte's theorem to answer the following questions.

   If $S \subseteq V$ is the "bad" set as per Tutte's theorem whereby $o(G' - S) > |S|$, then show that
   (i) S induces a complete subgraph in G',
   (ii) each connected component of $G' - S$ also introduces a complete subgraph in G', and (iii) the vertices in S are connected to all the vertices in G'.

   [Hint: Suppose the edge $\{u, v\}$ is absent in G' where $u, v \in S$. Then adding this edge to G' introduces a perfect matching in $G' + \{u, v\}$ but does not change violated Tutte's condition $o(G' + \{u, v\} - S) = o(G - S) > |S|$, a contradiction. Similar arguments apply for the connected components of $G' - S$, and also to edges between S and the components of $G' - S$.]

3. Show that 3-regular graphs with no cut edges have a 1-factor.

4. Show that every 3-regular graph with at most two cut edges has a 1-factor.

5. Show that interval graphs as well as their complements are perfect.

   [Given a set of closed intervals on a line, assign a vertex for each interval and an edge for every pair of intersecting intervals to define the interval graph. Sweep a line perpendicular the intervals.]

6. Show that the intersection graph of subtrees of a tree is a chordal graph.

7. Show that every minimal (by inclusion) vertex separator subset in a chordal graph is a complete graph.

   [Show that every component of $G - S$ for a minimal separator S, is adjacent to every vertex in S. Also, show that any two vertices u and v in S appear in a 4-cycle.]

8. Suppose a graph G has no 1-factor. Show that there must be a vertex x such that each edge adjacent to x is in some maximum matching.

[Let x be an unmatched vertex in a maximum matching F, and let y be a neighbour of x. If y is not covered by an edge e of F then $F + xy$ would be a larger matching. So, see $F - e + xy$ is also a maximum matching.]

9. Let G be a connected $2d$-regular graph with an even number of edges. Prove that G has a d-factor.

   [Consider an Euler trail L of G. Consider every second edge of the trail. This is possible as the number of edges is even. The selected edges for a d-factor.]

10. Show that the number of triangles in any simple graph of n vertices and m edges is at least $\frac{4m}{3n}(m - \frac{n^2}{4})$.

    [For any edge xy there are at least $d(x) + d(y) - n$ vertices adjacent to both x and y. So, this is also a lower bound on the number of triangles sitting on xy. A third of the sum of such estimates over all edges, lower bounds the number of triangles in the graph. This estimate is a third of $\sum(d(x))^2 - mn$, which is at least a third of n times the square of the average of vertex degrees minus mn by the Cauchy-Schwartz inequality.]