CS60047 Advanced Graph Theory	Date:29-09-23
Lecture 13	
Lecturer: Prof. S. P. Pal	*Scribe: Trishita Mukherjee*

13 Lecture Overview

In this lecture, we introduce some of the key theorems and propositions on inequalities and bounds of vertex colouring and edge colouring problems. We also introduce Helly property for intersecting subtrees of a tree. Formally, we sketch in detail the preliminary results of characterization of perfect graphs to develop a basis for the Perfect Graph Theorem. Also, proofs of applications of Hall's theorem, Tutte's theorem and Petersen's theorem of 1891 are established.

13.1 Problems on Bounds of Colouring and Helly property for intersecting subtrees of a tree

Proposition 13.1.1. *Is it true that $\chi(G) + \chi(\bar{G}) \leq n + 1$, where n is the number of vertices of the perfect graph G? Why? Does this hold for general graphs? Why?*

Solution: Yes, it is true that $\chi(G) + \chi(\bar{G}) \leq n + 1$, where n is the number of vertices of the perfect graph G.

For a perfect graph G, the chromatic number $\chi(G)$ is equal to the size of the largest clique in G, and the chromatic number of the complement graph $\bar{G}$ is equal to the size of the largest independent set in G.

However, this inequality does hold for general graphs as well. We can establish this proposition by induction for general graphs. The proof is sketched in the first part of **Proposition 13.1.2**.

Theorem 13.1.1. *Suppose $|V(G)| = n$ and $V(G)$ has a partition $\{ V_1, V_2, ..., V_k\}$ such that for each $1 \leq i < j \leq k$, there exists an $x \in V_i$ and a $y \in V_j$ which are non-adjacent. Then $\chi(G) \leq n - k + 1$.*

Proof. □

Proposition 13.1.2. *For an n-vertex graph G, show that $\chi(G) + \chi(\bar{G}) \leq n + 1$, and $\chi(G).\chi(\bar{G}) \geq n$.*

Proof. We prove $\chi(G) + \chi(\bar{G}) \leq n + 1$ by induction on the number of the vertices. It is easy to check the induction basis. Now suppose the inequality holds for all graphs with n vertices, we prove it for the graph G on $n+1$ vertices. Fix the vertex $v \in V(G)$ and let k be its degree in G, so the degree of v in $\bar{G}$ is $n-k$. Consider the graph $G-v$. Note that adding back v to $G-v$ does not increase the chromatic number if $\chi(G-v) > k$, since one can color v by an existing color different from the colors of its k neighbors; otherwise, it will increase the chromatic number by at most one. The same statement holds for $\bar{G}-v$ with the condition $\chi(\bar{G}-v) > n-k$. Therefore, if at least one of $\chi(G-v) > k$ and

$\chi(\bar{G}-v) > n-k$ holds, then applying induction hypothesis to $G-v$ will complete the proof:

$$\chi(G)+\chi(\bar{G}) \leq \chi(G-v)+\chi(\bar{G}-v)+1 \leq n+1 \quad (1)$$

Otherwise, we have $\chi(G-v) \leq k$ and $\chi(\bar{G}-v) \leq n-k$, which implies:

$$\chi(G)+\chi(\bar{G}) \leq \chi(G-v)+\chi(\bar{G}-v)+1 \leq k+n-k+1 = n+1 \quad (2)$$

This finishes the proof for $\chi(G)+\chi(\bar{G}) \leq n+1$.

We now prove that $\chi(G).\chi(\bar{G}) \geq n$.

□

Theorem 13.1.2. *Show that the edge chromatic number for a graph G is $\chi'(G) \leq 2\Delta-1$. Show also that for $\Delta(G) \geq 3$ this can be improved to $\chi'(G) \leq 2\Delta-2$.*

Proof. We define Line graphs to prove this theorem.

Definition 13.1.2.1. *For any graph G, the line graph of G, denoted $L(G)$, is the simple graph with vertex set $E(G)$, and adjacency determined by the rule that e, $f \in E(G)$ are adjacent vertices in $L(G)$ if they share an endpoint in G. Note that $\chi'(G) = \chi(L(G))$.*

If v is a vertex of degree $\Delta(G)$, then the edges of G incident with v form a clique in $L(G)$. Thus $\chi'(G) = \chi(L(G)) \geq \omega(L(G)) \geq \Delta(G)$. Every edge in G is adjacent to at most $2(\Delta(G)-1)$ other edges, so we have $\chi'(G) = \chi(L(G)) \leq \Delta(L(G))+1 \leq 2\Delta-1$.

Now, for the improvement of this bound, when $\Delta(G) \geq 3$, we can use Brooks theorem.

□

Theorem 13.1.3. *We wish to show that intersections of subtrees of a tree obey the Helly property, whereby the set $S = \{\, T|T \in \tau\}$ of pairwise intersecting subtrees of a tree T also has a non-empty intersection $\cap_{T\in S}T$. Complete the following argument for establishing this Helly property for intersecting subtrees of a tree [Gav96].*

Proof. To prove this theorem, let us define the Helly property for trees.

Definition 13.1.3.1. *Let T be a tree, and let τ be a finite family of subtrees of T such that each $S \in T$ has at least two vertices, and every pair of trees in τ has non-empty intersection; then $\cup\tau \neq \phi$. This is the Helly property of trees.*

We use induction on the number k of subtrees of an n-vertex tree τ . Conisder k subtrees $T_1, T_2, ..., T_k$ of τ which intersect pairwise. For the sake of contradiction we assume that they do not have a common intersection. However, by the induction hypothesis $T_1, T_2, ..., T_{k-1}$ intersect in say a subtree T_0. As T_k misses T_0 let us find a connecting path P from T_0 to T_k with a vertex $x \in P \cap T_k$ and a vertex y adjacent to x on P closer to T_0. Now $\tau - xy$ has connected components where the edge xy separates T_0 from T_k. ....

□

Theorem 13.1.4. *Show that $\chi'(G) \geq \Delta(G)$ for graph G. Show also that $\chi'(G) \geq \lceil \frac{E(G)}{\alpha'(G)} \rceil$, where E(G) is the number of edges of G.*

Proof. We need to show the trivial lower bound $\chi'(G) \geq \Delta(G)$ for graph G, where $\Delta(G)$ is the maximum degree of a vertex [Viz64].

Since, edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$ [Wes01]. This bound is almost tight for any graph.
Now we show that $\chi'(G) \geq \lceil \frac{E(G)}{\alpha'(G)} \rceil$, where $E(G)$ is the number of edges of G. In an edge coloring, the set of edges with any one color must all be non-adjacent to each other, so they form a matching. That is, a proper edge coloring is the same thing as a partition of the graph into disjoint matchings.

If the size of a maximum matching in a given graph is small, then many matchings will be needed in order to cover all of the edges of the graph. Expressed more formally, this reasoning implies that if a graph has $E(G)$ edges in total, and if at most α' edges may belong to a maximum matching, then every edge coloring of the graph must use at least $\lceil \frac{E(G)}{\alpha'(G)} \rceil$ different colors.

[For instance, the 16-vertex planar graph shown in the Fig. 1 has $E(G) = 24$ edges. In this graph, there can be no perfect matching; for, if the center vertex is matched, the remaining unmatched vertices may be grouped into three different connected components with four, five, and five vertices, and the components with an odd number of vertices cannot be perfectly matched. However, the graph has maximum matchings with seven edges, so $\alpha' = 7$. Therefore, the number of colors needed to edge-color the graph is at least $\lceil 24/7 \rceil$, and since the number of colors must be an integer it is at least four.]

□

Theorem 13.1.5. *Show that $\chi'(G) = \Delta(G)$ for bipartite graph G [Konig 1916].*

Proof. We apply induction on G. For $G = 0$ the assertion holds. Now assume that $G \geq 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a Δ-edge-colouring of $G - xy$ by the induction hypothesis. Let us refer to the edges coloured α as α-edges, etc.

In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges. Hence there are $\alpha, \beta \in 1, ..., \Delta$ such that x is not incident with an α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge. Let us extend this edge to a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice, W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G. We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G. Therefore, $\chi'(G) = \Delta(G)$ for bipartite graph G [Die00, Wes01].

□

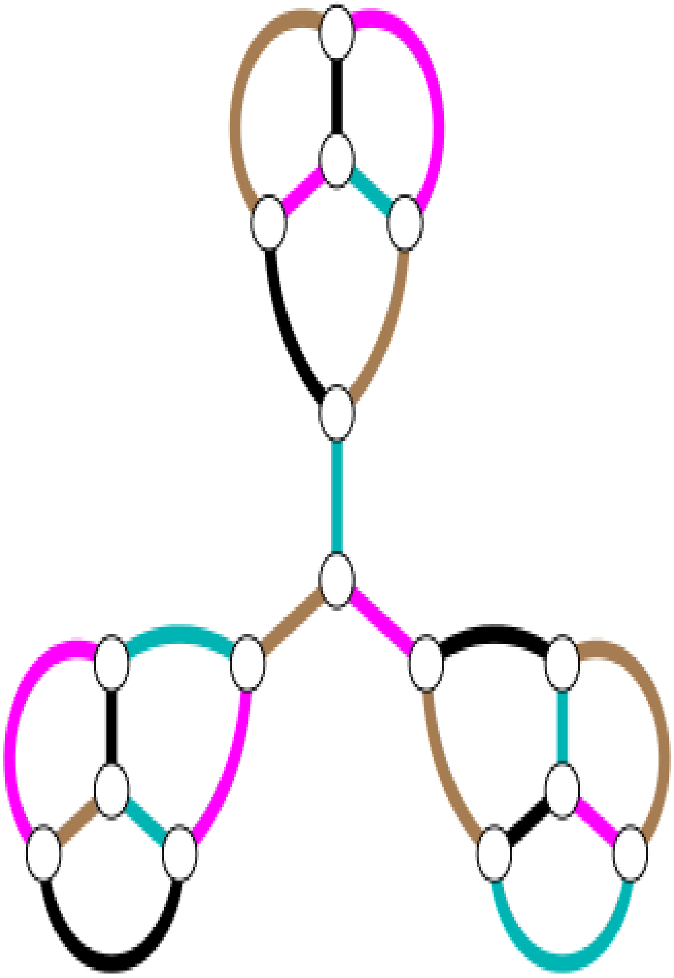

Figure 1: 2-planar-3-regular graph Edge coloring

13.2 Characterization of Perfect Graphs

The result that a graph is perfect if and only if its complement is perfect is called the Perfect Graph Theorem. There are several proofs of it in existence; we will focus on the way it was proven originally, by Lov´asz and Fulkerson in 1970. There are some preliminary results proved to establish the Perfect Graph Theorem.

Lemma 13.2.1. *Generating perfect graphs by extension of a vertex.*

Proof. The graph obtained by expanding a vertex of a perfect graph is also perfect. A vertex x in a graph G is expanded by adding a new vertex x' and connecting x' to x and all neighbours of x in G, thus obtaining the expanded graph G'.

This result is established using induction on the number of vertices. Coming to expanding G at vertex x, introducing edge xx' by adding the new vertex x', we get graph G', where x' connects to all neighbours of x in G. We show that G' is perfect if G is so. We use induction, with the basis case of expansion of K_1 to K_2, which are both perfect. Now G is perfect so for G' to be shown perfect we need to only show $\chi(G') \leq \omega(G')$. This is so because every proper induced subgraph H of G' is either isomorphic to some induced subgraph of G (and therefore perfect with $\chi(H) \leq \omega(H)$), or created from a proper induced subgraph of G by the expansion of G.

If it is the second case above then the induced subgraph H of G' must have x', and a proper induced subgraph K of G, with or without x. If x is not there then $H = K + \{x'\}$ is just like an isomorph of a proper induced subgraph $K + x$ of G where x' acts just like x. Otherwise we have the non-trivial case where both x and x' are in H. In this case the subgraph H of G' is perfect, by the induction hypothesis and the expansion construction. This is because we can use induction for showing that the extension of a proper induced perfect subgraph of G at a vertex yields a perfect graph H, even if it has both x and x'. So now we have shown that in all the possible cases for a proper subgraph H of G', H is indeed perfect and therefore has a $\omega(H)$ coloring. Now we only need show that $\chi(G') \leq \omega(G')$. Let $w = \omega(G)$, then $\omega(G')$ is either w or $w + 1$. The easier case is when the maximum clique size is $w + 1$. Then,

$$\chi(G') \leq \chi(G) + 1 = w + 1 = \omega(G') \tag{3}$$

as we may need just one more colour and G is perfect.

However if $\omega(G') = w$, then note that x is not in any K_w of G, as otherwise, together with x', that would yield a K_{w+1} in G', a contradiction to $\omega(G')$ being w. Observe that our definition of extension of G to G' at x by x' now helps us in using this trump card. Now G being perfect we color G with $\omega(G)$ colors. But x misses all K_w of G, though the color class X of x would not miss any K_w of G. See the graph $H = G - (X\backslash\{x\})$, which misses the color class X but not x, and has $\omega(H) < w$. By the induction hypothesis (H being a proper induced subgraph of G, and thus being perfect), we can color H with $w - 1$ colors. Now X is an independent set but observe from the expansion construction of x' that $(X\backslash\{x\}) \cup \{x'\}$ is also an independent set as x and x' play similar connectivity roles, and this set is exactly all vertices in G' but not those in H by definition of H, X and x'. So the $(w - 1)$-coloring of H can be extended to a w-coloring of G'.

□

Proposition 13.2.1. *Show that a graph G is perfect if and only if it has the property that every induced subgraph H contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$.*

Proof. As this is an if and only if statement, we must prove both directions. Let G be a perfect graph. Because any induced subgraph of G is perfect, it suffices to find A such that $\omega(G - A) < \omega(G)$. Doing this is trivial, however: just take any $\chi(G) = \omega(G)$ coloring of G, and let A be one of the color classes used in this coloring. This set is independent by definition; as well, because we have removed one color, $G - A$ satisfies $\omega(G - A) = \chi(G - A) < \chi(G) = \omega(G)$. Thus, we have proven this direction of our claim.

Now, take a graph G such that every induced subgraph $H \subset G$ has an independent set A_H intersecting every clique in H of maximal order. We seek to show that such a graph is perfect, and do so by inducting on $\omega(G)$. As the only graphs with $\omega(G) = 1$ are the edgeless graphs, there is nothing to prove here; so we assume that $\omega(G) = n$ and that we've proven our result for all values of $n' < n$.

Let H be any induced subgraph of G, and let A be an independent set of vertices in H such that $\omega(H - A) < \omega(A)$. By our inductive hypothesis, $H - A$ is perfect, and therefore

$\chi(H-A) = \omega(H-A)$; so we can color $H-A$ with $\omega(H-A)$-many colors. Take any such coloring, and extend it to a coloring of H by coloring A some other, new color; this gives us a coloring of H with $\omega(H-A)+1$ many colors. Because $\omega(H-A) < \omega(H)$, this means that our coloring of H uses $\leq \omega(H)$ many colors: i.e., that $\omega(H) \leq \chi(H)$. But this means that $\omega(H) = \chi(H)$. As this holds for every subgraph H of G, we've proven that G is perfect, as claimed [Lov72a].

□

Lemma 13.2.2. *Show that the extention by arbitrary perfect graphs at each of the vertices of a given perfect graph G, again gives a perfect graph [Lov72b].*

13.3 Applications of Hall's and Tutte's Theorems

Theorem 13.3.1. [1] ***Hall's Theorem - P. Hall [1935]:****A Bipartite graph $G(X \cup Y, E)$ has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

Corollary 13.3.1.1. *Let G be a connected $2d$-regular graph with an even number of edges. Prove that G has a d-factor.* (*Hw2 Problem 9*)

Theorem 13.3.2. [2] ***Tutte's Theorem- Tutte [1947]*** *A graph G has 1-factor if and only if $o(G-S) \leq |S|$ for all $S \subseteq V(G)$.*

Corollary 13.3.2.1. *Show that 3-regular graphs with no cut edges have a 1-factor.*

Corollary 13.3.2.2. *Show that every 3-regular graph with at most two cut edges has a 1-factor.* (*The proof of it contains some definitions and notations from* [*Wes11*])

Proof. Every 3-regular graph $G = (V, E)$ has an even number of vertices as $\sum\limits_{v \in V(G)} deg(v) = 3|V(G)| = 2|E(G)|$ where RHS is even $\implies$ LHS is even $\implies$ $|V(G)|$ is even. For the sake of contradiction, we assume that G does not have 1-factor which is a result of G's violation of the necessary Tutte's condition for the existence of 1-factor in G. In such a G, we must have a deficient set $S \subseteq V(G)$ such that $o(G-S) \geq |S|$ where by $o(H)$ we mean the number of odd components(connected components with an odd number of vertices) in graph H. Let us define the deficiency of set S, $\text{def}(S) = o(G-S) - |S|$. Saturating all vertices in an odd component of $G-S$ by a matching in G requires matching one of its vertices with a vertex of S. A Tutte set is a vertex subset with positive deficiency. Let $\text{def}(G) = \max_{S \subseteq V(G)} \text{def}(S)$. In an n vertex graph, the maximum matching leaves $\text{def}(G)$ number of vertices unsaturated. Since $|V(G)|$ is even in G and any matching saturates an even number of vertices always, in case of a maximum matching in such a graph G, $\text{def}(G) \geqslant 2$. So, $\exists S \subseteq V(G)$, such that $o(G-S) - |S| \geq 2$. In $G-S$, the odd components have odd number of vertices each and in G each such vertex has degree 3. So, the sum of the degrees of vertices of any such odd component in $G-S$ is odd in G. Since the sum of the degrees within the odd component is even as its just twice the number of edges

[1]Note*: The proof of Hall's theorem is already done in previous lectures, therefore only the theorem is mentioned to establish the proof of the corollary.

[2]Note*: The proof of Tutte's theorem is already done in previous lectures, therefore only the theorem is mentioned to establish the proofs of the corollaries.

within that component which is an even number, the number of edges between S and such a component is odd. If there is only 1 edge between S and an odd component, the edge is termed as the cut edge and according to G, it has at most two cut edges. Hence, only two odd components have 1 edge each between S and themselves. Every other odd components have at least 3 edges between S and themselves. Hence, there are at least $3(o(G-S)-2)+2$ edges between S and $G-S$ which results in at least $3|S|+2$ edges. This is contradictory to the fact that vertices of the set S can have at most $3|S|$ edges with other endpoint of each such edge is in $G-S$ since G is 3-regular. So, G has a 1-factor. $\square$

Theorem 13.3.3. *Show that every regular graph of even degree has a 2-factor (Petersen [1891]).*

References

[Die00] Reinhard Diestel. Graph theory, 2000.

[Gav96] Fănică Gavril. Intersection graphs of helly families of subtrees. *Discrete Applied Mathematics*, 66(1):45–56, 1996.

[Lov72a] László Lovász. A characterization of perfect graphs. *Journal of Combinatorial Theory, Series B*, 13(2):95–98, 1972.

[Lov72b] László Lovász. Normal hypergraphs and the perfect graph conjecture. *discrete Mathematics*, 2(3):253–267, 1972.

[Viz64] Vadim G Vizing. On an estimate of the chromatic class of a p-graph. *Diskret analiz*, 3:25–30, 1964.

[Wes01] Douglas West. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[Wes11] Douglas B. West. A short proof of the berge–tutte formula and the gallai–edmonds structure theorem. *European Journal of Combinatorics*, 32(5):674–676, 2011.