Balanced Parentheses

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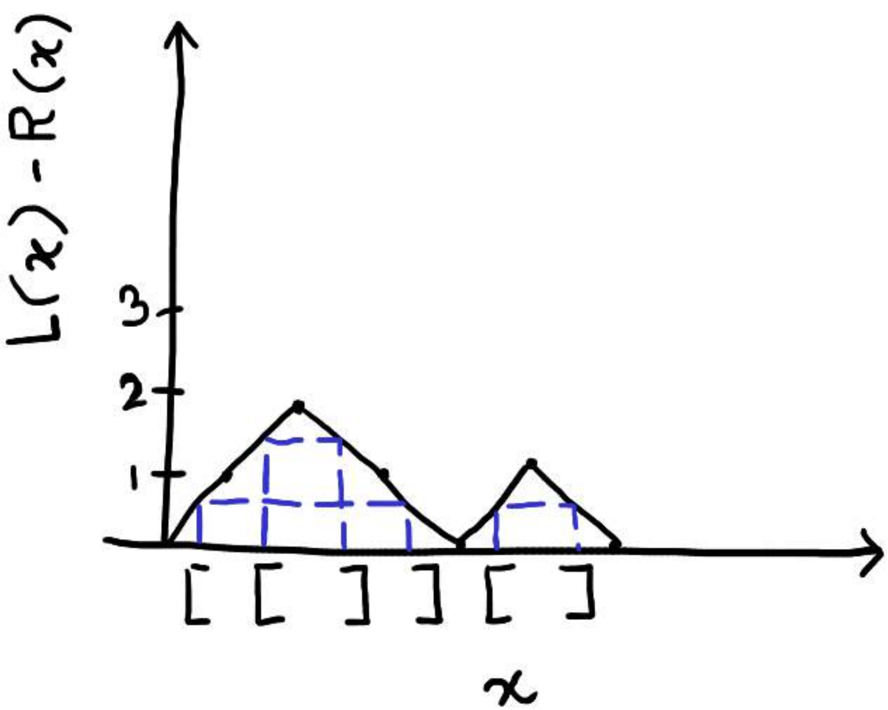
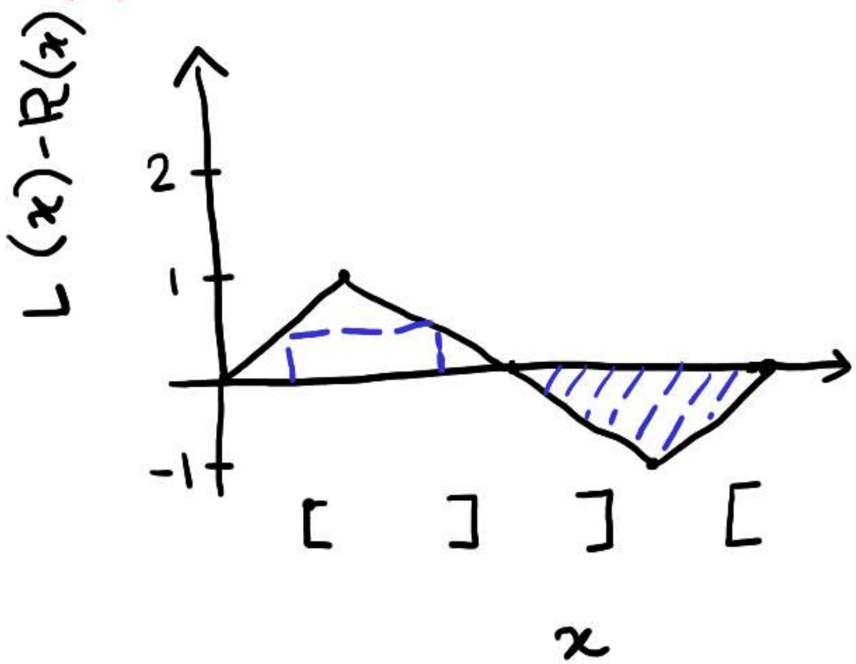
- Necessary conditions: A string x of parentheses is *balanced* iff:
  (i) $L(x) = R(x)$,
  (ii) for all prefixes y of x, $L(y) \geq R(y)$. - A right parenthesis can only match to a left parenthesis to its left.

Sufficient Conditions for Balance

- The above conditions are sufficient:
  Look at the graph of $L(x) - R(x)$ v x.





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- Proof:
  ($\Rightarrow$) If $S \rightarrow_G^* x$ then x satisfies (i) and (ii).
- Induction on length of the derivation of x.
  In fact, we show that for any $\alpha \in (N \cup \Sigma)^*$, if $S \rightarrow_G^* \alpha$, then α satisfies (i) and (ii).
  In fact, induction on length of derivation of α.

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- Induction step: $S \rightarrow^n_G \beta \rightarrow^1_G \alpha$.
- By IH, β satisfies (i) and (ii).

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- Similar argument for $S \to SS$.
- $S \to [S]$: Then $\beta = \beta_1 S \beta_2$ and $\alpha = \beta_1 [S] \beta_2$.
- Condition (i): $L(\alpha) = L(\beta) + 1$
  $= R(\beta) + 1$ (IH on β and (i))
  $= R(\alpha)$

- Condition (ii): Want to show that for any prefix γ of $\alpha = \beta_1[S]\beta_2$, $L(\gamma) \geq R(\gamma)$.

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- If γ is a prefix of β_1, then it is a prefix of β - so done by IH.
- If γ is a prefix of $\beta_1[S$ but not β_1, then
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- If $\gamma = \beta_1[S]\delta$ where δ is a prefix of β_2, then
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- Thus (ii) also holds for α and this concludes the proof of ($\Rightarrow$): If $S \rightarrow_G^* \alpha$, then α is balanced [In particular, when α is a sentence x].

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- By IH $S \rightarrow^*_G y$ and $S \rightarrow^*_G z$. Then
  $S \rightarrow^1_G SS \rightarrow^*_G yS \rightarrow^*_G yz = x$.

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- Thus, grammar $S \rightarrow [S]|SS|\epsilon$ generates exactly the set of strings satisfying the 2 balanced parentheses conditions.